

EMERGENT GEOMETRY AND MIRROR SYMMETRY OF A POINT

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Dedicated to the memory of Professor Linsheng Yin

ABSTRACT. By considering the partition function of the topological 2D gravity, a conformal field theory on the Airy curve emerges as the mirror theory of Gromov-Witten theory of a point. In particular, a formula for bosonic n -point functions in terms of fermionic 2-point function for this theory is derived.

1. INTRODUCTION

Witten Conjecture/Kontsevich Theory and mirror symmetry are two main themes in Gromov-Witten theory and its many recent variants. A combination of them lead us to study the mirror symmetry of a point in [30]. This involves the Witten-Kontsevich tau-function τ_{WK} which encodes some intersection numbers on Deligne-Mumford compactifications of moduli spaces of algebraic curves. As a result τ_{WK} is a formal power series in infinitely many variables t_0, t_1, t_2, \dots , where t_0 is the coupling constant of a primary field and it gives a linear coordinate on the small phase space, and t_n ($n \geq 1$) are the coupling constants of some descendant fields and they give linear coordinates on the big phase space. By emergent geometry we mean the geometric structures that naturally appear when one considers such situations with infinite degrees of freedom, which however cannot be seen easily when one restricts to the finite-dimensional subspaces.

Let us make an analogy between the study of Gromov-Witten like theories and statistical physics. In the latter one has a system consisting of large amount of particles, and one can roughly approximate the system as having an infinite degrees of freedom. In the former, one has a finite-dimensional space called the small phase space whose elements in a basis will be referred to as primary fields, each of which has infinitely many gravitational descendents. The primaries and their descendents generate the infinite-dimensional big phase space. The partition function and the free energy are formal power series formally regarded as defined on the big phase space. So for both problems one

has an infinite degree of freedom. The genus zero part of the free energy for a Gromov-Witten like theory restricts to the small phase space to give it a geometric structure of a Frobenius manifold [7]. A major problem is to reconstruct from this structure the whole the theory [10, 8]. Of course one does not expect to achieve this goal from the Frobenius manifold structure alone without knowing in advance some properties satisfied by the free energy on the big phase space. So in the theory of Frobenius manifold, even though one is dealing with a reconstruction problem, some considerations from the emergent point of view is necessary, e.g., one has to assume the existence of a tau-function which satisfies the loop equation [8].

As advocated in an article by Anderson [3] and a book by Laughlin [15], there are the reduction/reconstruction and the emergence approaches to science. In the former approach collective behavior of large quantity of individual particles is derived from the fundamental laws obeyed by each individual particle, in the latter approach there are fundamental laws at each level of complexity [3], and it is even possible that all the fundamental laws for individual particles have their origins in their collective behavior [15, Preface, XV].

The analogy with statistical physics suggests that for Gromov-Witten like theories, one might take an emergent approach to derive everything from universal properties of the free energy in all genera on the big phase space. One is then led to understand Frobenius manifold structure, integrable hierarchy, W-constraints, spectral curves, etc., as all emerge from general properties of the free energy on the big space. We will take topological recursion relations as the universal relations satisfied by Gromov-Witten like theories, and show how other structures emerge naturally as one develops the theories based on them. We call the procedures of doing this the emergent geometry of the corresponding theory. In this paper we will focus on the theory of topological 2D gravity whose partition function is Witten-Kontsevich tau-function. Generalization to other theories related to general Frobenius manifolds will be reported in a separate paper [32].

The function τ_{WK} has the following two equivalent characterizations [25, 14]: (a) It is a tau-function of the KdV hierarchy and it satisfies the puncture equation; (b) It satisfies the Virasoro constraints. We can now understand them from an emergent point of view. In an earlier paper [30] the author proposed a notion of a quantum deformation theory to study mirror symmetry. Roughly speaking, this means a deformation theory which encodes the information of free energy in all genera on the big phase space. For such a theory, the moduli space is infinite-dimensional so as to encode all information of gravitational descendants

in genus zero, and furthermore, it admits a natural quantization from which one can produce constraints that determines the free energy in all genera. We will recall the case of topological 2D gravity and mirror symmetry of a point in §2 below and reformulate it from the point of view of emergent geometry.

Note in [30] we have understood the appearance of Virasoro constraints from the quantum deformation theory of the Airy curve and have not addressed the appearance of KdV hierarchy from this point of view. That is exactly what we will do in this paper. We will show that quantum deformation theory naturally leads us to a version of noncommutative deformation theory and Kyoto school's approach to integrable hierarchies. We will also discuss the emergence of Airy functions and its appearance in the proof of Witten Conjecture [14, 20] and some related recent works on explicit formula for τ_{WK} [29, 4] and n -point functions [6, 31].

As a complement of the results of [30], we now have a more complete formulation of the mirror symmetry of a point. The mirror of a point is the following emergent conformal field theory living near the infinity of the Airy curve $y = \frac{1}{2}x^2$, it is determined by a vector in fermionic Fock space:

$$(1) \quad |W\rangle = e^A |0\rangle \in \mathcal{F}^{(0)},$$

where the operator A of the form

$$A = \sum_{m,n \geq 0} A_{m,n} \psi_{-m-1/2} \psi_{-n-1/2}^*$$

is specified by [29, 4], and can be found as follows. Let

$$(2) \quad a(x) = \sum_{m=0}^{\infty} \frac{(6m-1)!!}{6^{2m}(2m)!} x^{-3m},$$

$$(3) \quad b(y) = - \sum_{m=0}^{\infty} \frac{(6m-1)!!}{6^{2m}(2m)!} \frac{6m+1}{6m-1} y^{-3m+1}.$$

Then $A(x, y) = \sum_{m,n \geq 0} A_{n,m} x^{-m-1} y^{-n-1}$ is given by:

$$(4) \quad A(x, y) = \frac{a(-x) \cdot b(y) - a(y)b(-x)}{x^2 - y^2} - \frac{1}{x - y}.$$

Furthermore, the n -point functions at $t_0 = t_1 = \dots = 0$ are given as follows. For $F = \log Z_{WK}$, $T_{2n+1} = (2n+1)!!t_n$,

$$(5) \quad \sum_{j_1, \dots, j_n \geq 1} \frac{\partial^n F}{\partial T_{j_1} \cdots \partial T_{j_n}} \Big|_{\mathbf{T}=0} \xi_1^{-j_1-1} \cdots \xi_n^{-j_n-1} \\ = (-1)^{n-1} \sum_{n\text{-cycles}} \prod_{i=1}^n \hat{A}(\xi_{\sigma(i)}, \xi_{\sigma(i+1)}),$$

where

$$(6) \quad \hat{A}(\xi_i, \xi_j) = \begin{cases} A(\xi_i, \xi_j), & \text{if } i = j, \\ \frac{1}{\xi_i - \xi_j} + A(\xi_i, \xi_j), & \text{if } i \neq j. \end{cases}$$

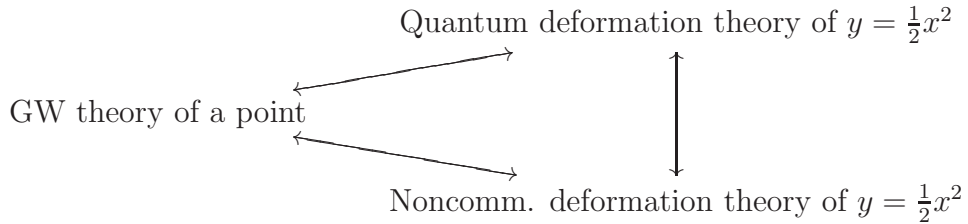
In summary, we have the following picture for mirror symmetry of a point:

$$\text{GW theory of a point} \Leftrightarrow \text{Conformal field theory on } y = \frac{1}{2}x^2$$

The series $a(x)$ and $b(x)$ are related to the asymptotic expansions of the Airy function $\text{Ai}(x)$ and its derivative $\text{Ai}'(x)$ respectively. The Airy function is a solution of the Airy equation which can be obtained by a quantization of the Airy curve. In quantum deformation theory of the Airy curve [30] we consider the generalization of the miniversal deformation

$$(7) \quad y = \frac{1}{2}x^2 + t_0,$$

specified by the Virasoro constraints, in this paper we consider the deformation of its quantization $\frac{1}{2}\partial_{t_0}^2 + t_0$ to $\frac{1}{2}\partial_{t_0}^2 + u(\mathbf{T})$, specified by the KdV hierarchy. We summarize these in the following picture:



We will report on generalizations of the above pictures in a separate paper [32].

The other Sections of this paper are arranged as follows. In Section 2 we explain the emergence of quantum deformation theory from topological recursion relations. In Section 3 we explain how quantum deformation theory leads naturally to Sato's Grassmannian, KP hierarchy and a conformal field theory. In Section 4 we use fermionic one-point

functions to understand the KP hierarchy and its tau-functions, and in Section 5 we establish a formula for bosonic n -point functions in terms of fermionic two-point function. In Section 6 we explain the emergence of Airy function and its applications.

In this work we have presented an approach to KP hierarchy from a conformal field theory point of view, and treat the applications to KdV hierarchy as a special case. This is because we are preparing for a uniform treatment [32] of all Witten r -spin curve intersection numbers, and other cases related to semisimple Frobenius manifolds. See e.g. [5] for applications to the case of r -spin curves.

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2. EMERGENT REFORMULATION OF QUANTUM DEFORMATION THEORY OF THE AIRY CURVE

In this Section we first briefly recall the quantum deformation theory of the Airy curve as developed in [30], then we present an emergent interpretation by topological recursion relations.

2.1. Quantum deformation theory of the Airy curve. We first use the Virasoro constraints to compute the genus zero one-point function on the small phase space:

$$(8) \quad \frac{\partial F_0}{\partial t_n}(t_0, 0, \dots) = \frac{1}{(n+2)!} t_0^{n+2}.$$

Next we note the Puiseux series:

$$(9) \quad x = f - \frac{t_0}{f} - \sum_{n \geq 0} (2n+1)!! \frac{\partial F_0}{\partial t_n}(t_0, 0, \dots) \cdot f^{-2n-3},$$

where $f^2 = 2y$, leads to the miniversal deformation of the Airy curve

$$(10) \quad y = \frac{1}{2}x^2 + t_0.$$

When $t_0 = 0$, one gets the Airy curve which is the spectral curve for Eynard-Orantin topological recursion for topological 2D gravity [28]. Next we construct a special deformation of the Airy curve of the following form:

$$(11) \quad x(f) := f - \sum_{n \geq 0} \frac{t_n}{(2n-1)!!} f^{2n-1} - \sum_{n \geq 0} (2n+1)!! \frac{\partial F_0}{\partial t_n}(\mathbf{t}) \cdot f^{-2n-3}.$$

We have proved that it is uniquely characterized by the following property:

$$(12) \quad (x(f)^2)_- = 0,$$

and this is equivalent to Virasoro constraints in genus zero. To extend the picture to arbitrary genera, the following quantization is used. We endow the space of series of the form

$$(13) \quad \sum_{n=0}^{\infty} (2n+1) \tilde{u}_n z^{(2n-1)/2} + \sum_{n=0}^{\infty} \tilde{v}_n z^{-(2n+3)/2}$$

the following symplectic structure:

$$(14) \quad \omega = \sum_{n=0}^{\infty} d\tilde{u}_n \wedge d\tilde{v}_n.$$

Consider the canonical quantization:

$$(15) \quad \hat{u}_n = \frac{t_n - \delta_{n,1}}{(2n+1)!!}, \quad \hat{v}_n = (2n+1)!! \frac{\partial}{\partial t_n}.$$

Corresponding to the field x , we consider the following field of operators on the Airy curve:

$$(16) \quad \hat{x}(z) = - \sum_{m \in \mathbb{Z}} \beta_{-(2m+1)} z^{m-1/2} = - \sum_{m \in \mathbb{Z}} \beta_{2m+1} z^{-m-3/2}$$

where $f = z^{1/2}$ and the operators β_{2k+1} are defined by:

$$(17) \quad \beta_{-(2k+1)} = (2k+1) \frac{t_k - \delta_{k,1}}{(2k+1)!!}, \quad \beta_{2k+1} = (2k+1)!! \frac{\partial}{\partial t_k}.$$

We define a notion of regularized products $\hat{x}(z)^{\odot n}$ and show that the DVV Virasoro constraints for Witten-Kontsevich tau-function is just the following equation:

$$(18) \quad (\hat{x}(z)^{\odot 2})_- Z_{WK} = 0.$$

2.2. Topological recursion relations in genus zero for topological 2D gravity. Recall for topological 2D gravity, the n -point correlators are defined by

$$(19) \quad \langle \tau_{m_1} \cdots \tau_{m_n} \rangle_g := \int_{\mathcal{M}_{g,n}} \psi_1^{m_1} \cdots \psi_n^{m_n}.$$

The correlators in genus zero satisfy the topological recursion relations [25, (2.26)]:

$$(20) \quad \langle \tau_{m_1} \cdots \tau_{m_n} \rangle_0 = \sum_{X \amalg Y = \{2, \dots, n-2\}} \langle \tau_{m_1-1} \prod_{j \in X} \tau_{m_j} \cdot \tau_0 \rangle_0 \cdot \langle \tau_0 \prod_{k \in Y} \tau_{m_k} \cdot \tau_{m_{n-1}} \tau_{m_n} \rangle_0.$$

This relation reduces the calculations of the correlators in genus zero to the initial value:

$$(21) \quad \langle \tau_0^3 \rangle_0 = 1.$$

2.3. Emergence of a Froebnius manifold. In particular, one gets:

$$(22) \quad \langle \tau_n \tau_0^{n+2} \rangle_0 = 1.$$

Of course one can also get this by the puncture equation:

$$(23) \quad \langle \tau_0 \prod_{i=1}^n \tau_{m_i} \rangle_0 = \sum_{j=1}^n \langle \prod_{i=1}^n \tau_{m_i - \delta_{ij}} \rangle_0.$$

But the reason why we use the TRR in genus zero is that one can deduce from it the fact that the small phase space with coordinate t_0 and potential function $F_0(t_0) = \frac{t_0^3}{3!}$ is a one-dimensional Frobenius manifold. This is the route to the emergence of the Frobenius manifold structure that we will take when we make the generalizations in [32].

2.4. Emergence of the Airy curve and its versal deformation. By (22) we have

$$(24) \quad \frac{\partial F_0}{\partial t_n}(t_0) := \frac{\partial F_0}{\partial t_n}(\mathbf{t}) \Big|_{\mathbf{t}=(t_0, 0, \dots)} = \frac{t_0^{n+2}}{(n+2)!}.$$

Therefore, one can bypass the use of Virasoro constraints in [30] to get this result. Consider the generating series

$$(25) \quad t_0(z) = t_0 + \sum_{n \geq 0} \frac{\partial F_0}{\partial t_n}(t_0) \cdot z^{n+1}.$$

Take a Laplace transform of $t_0(z)$:

$$(26) \quad \begin{aligned} \tilde{t}_0(y) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{z}} e^{-yz/2} \cdot t_0(z) dz \\ &= \frac{t_0}{y^{1/2}} + \sum_{n \geq 0} \frac{(2n+1)!!}{y^{n+3/2}} \frac{\partial F_0}{\partial t_n}(t_0). \end{aligned}$$

Now define a Puiseux series $x(y)$ by:

$$(27) \quad x(y) = \tilde{t}_0(y) - y^{1/2}.$$

More concretely,

$$(28) \quad x(y) = -y^{1/2} + \sum_{n \geq 0} \frac{(2n-1)!!}{(n+1)!} t_0^{n+1} y^{-n-1/2} = -(y - 2t_0)^{1/2}.$$

This is equivalent to:

$$(29) \quad y = x^2 + 2t_0.$$

This is the versal deformation of the Airy curve:

$$(30) \quad y = x^2.$$

2.5. Emergent interpretations by ghost variables. Now we understand the extra term t_0 on the right-hand side of (25), the extra term $-y^{1/2}$ and the special deformation from an emergent point of view. As pointed out in [28], the following convention for one-point and two-point genus zero correlators plays a crucial role in understanding the topological 2D gravity even though they are not well defined geometrically:

$$(31) \quad \langle \tau_n \rangle_0 = \delta_{n,-2},$$

$$(32) \quad \langle \tau_k \tau_{-k-1} \rangle_0 = (-1)^k,$$

so that

$$(33) \quad \sum_{n \in \mathbb{Z}} \langle \tau_n \rangle_0 x^n = \frac{1}{x^2},$$

$$(34) \quad \sum_{k \geq 0} \langle \tau_k \tau_{-k-1} \rangle_0 x^k y^{-k-1} = \sum_{k \geq 0} (-1)^k x^k y^{-k-1} = \frac{1}{x+y}.$$

These conventions are natural since it is well-known that for $n \geq 3$,

$$(35) \quad \sum_{m_1, \dots, m_n \geq 0} \langle \tau_{m_1} \cdots \tau_{m_n} \rangle_0 x_1^{m_1} \cdots x_n^{m_n} = (x_1 + \cdots + x_n)^{n-3}.$$

In [28] it was shown that these conventions determine the equation of the spectral curve and the Bergman kernel for Eynard-Orantin recursions for topological 2D gravity.

One formally adds

$$(36) \quad t_{-2} + \sum_{n \geq 0} (-1)^n t_n t_{-n-1} = \sum_{n \geq 0} (-1)^n (t_n - \delta_{n,1}) t_{-n-1}$$

to the genus zero free energy F_0 to get the full genus zero free energy:

$$(37) \quad \tilde{F}_0 = F_0 + \sum_{n \geq 0} (-1)^n (t_n - \delta_{n,1}) t_{-n-1}.$$

The variables t_{-1}, t_{-2}, \dots will be referred to as the ghost variables. The space that include also these ghost variables will be called the full phase space. Considered the augmented generating series of genus zero one-point function on the full phase space:

$$(38) \quad \sum_{n \in \mathbb{Z}} \frac{\partial \tilde{F}_0}{\partial t_n} \cdot z^{n+1} = \sum_{n \geq 0} (-1)^n (t_n - \delta_{n-1}) z^{-n} + \sum_{n \geq 0} \left(\frac{\partial F_0}{\partial t_n}(\mathbf{t}) + (-1)^n t_{-n-1} \right) \cdot z^{n+1}.$$

When restricted to the small phase space:

$$(39) \quad \sum_{n \in \mathbb{Z}} \frac{\partial \tilde{F}_0}{\partial t_n} \cdot z^{n+1} \Big|_{t_n = \delta_{n,0} t_0} = t_0(z) + z^{-1}.$$

So $t_0(z) + z^{-1}$ is the augmented genus zero one point function on the small phase space:

$$(40) \quad t_0(z) + z^{-1} = \langle \langle \frac{1}{z - c_1} \rangle \rangle_0(t_0).$$

In the theory of Frobenius manifolds [8], $t_0(z)$ is the deformed flat coordinate for Dubrovin connection, and the Laplace transform \tilde{t}_0 is the period of the Gauss-Manin system of the Frobenius manifold. Such interpretation will be used in [32] to make generalizations to more general Frobenius manifolds.

Now we explain the extra term $-y^{1/2}$ on the right-hand side of (27). For $a \geq 0$, one has the following formula for Laplace transform:

$$(41) \quad \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{z}} e^{-yz/2} \cdot z^a dz = \frac{\Gamma(a + 1/2)}{\sqrt{2\pi}} \frac{2^{a+1/2}}{y^{a+1/2}}.$$

We use this to define the Laplace transform of z^{-n} to be

$$(42) \quad \frac{\Gamma(-n + 1/2)}{\sqrt{2\pi}} \frac{2^{-n+1/2}}{y^{-n+1/2}} = \frac{(-1)^n}{(2n-1)!!} y^{n-1/2}.$$

It follows that

$$(43) \quad \begin{aligned} & \sum_{n \in \mathbb{Z}} \frac{\partial \tilde{F}_0}{\partial t_n} \cdot \frac{1}{\sqrt{2\pi}} \int_0^\infty z^{n+1} e^{-yz/2} dz \\ &= \sum_{n \geq 0} \frac{t_n - \delta_{n-1}}{(2n-1)!!} y^{n-1/2} \\ &+ \sum_{n \geq 0} \left(\frac{\partial F_0}{\partial t_n}(\mathbf{t}) + (-1)^n t_{-n-1} \right) \cdot (2n+1)!! y^{-n-3/2}. \end{aligned}$$

By restricting to the big phase space, this provides an interpretation from emergent point of view of the special deformation introduced in the quantum deformation theory of the Airy curve [30]:

$$(44) \quad p(y) = \sum_{n \geq 0} \frac{t_n - \delta_{n,1}}{(2n-1)!!} y^{n-1/2} + \sum_{n \geq 0} (2n+1)!! \frac{\partial F_0}{\partial t_n}(\mathbf{t}) \cdot y^{-n-3/2}.$$

A further restriction to the small phase space then yields (27).

2.6. Genus zero two-point function. Let us generalize the above treatment to genus zero two-point function on the full phase space:

$$(45) \quad \sum_{n_1, n_2 \in \mathbb{Z}} \frac{\partial^2 \tilde{F}_0}{\partial t_{n_1} \partial t_{n_2}} \cdot z_1^{n_1+1} z_2^{n_2+1} = \sum_{n \geq 0} (-1)^n (z_1^{-n} z_2^{n+1} + z_2^{-n} z_1^{n+1}) \\ + \sum_{n_1, n_2 \geq 0} \frac{\partial^2 F_0}{\partial t_{n_1} \partial t_{n_2}}(\mathbf{t}) \cdot z_1^{n_1+1} z_2^{n_2+1}.$$

When restricted to the small phase space:

$$(46) \quad \frac{\partial^2 F_0}{\partial t_k \partial t_l}(t_0) = \binom{k+l}{k} \frac{t_0^{k+l+1}}{(k+l+1)!},$$

and so

$$(47) \quad \sum_{n_1, n_2 \in \mathbb{Z}} \frac{\partial^2 \tilde{F}_0}{\partial t_{n_1} \partial t_{n_2}} \cdot z_1^{n_1+1} z_2^{n_2+1} \Big|_{t_n = \delta_{n0} t_0} \\ = \sum_{n \geq 0} (-1)^n (z_1^{-n} z_2^{n+1} + z_2^{-n} z_1^{n+1}) \\ + \sum_{k, l \geq 0} \binom{k+l}{k} \frac{t_0^{k+l+1}}{(k+l+1)!} \cdot z_1^{k+1} z_2^{l+1}.$$

After taking the summation,

$$(48) \quad \sum_{n_1, n_2 \in \mathbb{Z}} \frac{\partial^2 \tilde{F}_0}{\partial t_{n_1} \partial t_{n_2}} \cdot z_1^{n_1+1} z_2^{n_2+1} \Big|_{t_n = \delta_{n0} t_0} \\ = i_{z_1, z_2} \frac{1}{z_1 + z_2} + i_{z_2, z_1} \frac{1}{z_1 + z_2} + \frac{1}{z_1 + z_2} (e^{(z_1+z_2)t_0} - 1),$$

where

$$(49) \quad i_{z_1, z_2} \frac{1}{z_1 + z_2} = \sum_{n \geq 0} (-1)^n z_1^{-n} z_2^{n+1}, \\ i_{z_2, z_1} \frac{1}{z_1 + z_2} = \sum_{n \geq 0} (-1)^n z_2^{-n} z_1^{n+1}.$$

Take Laplace transform:

$$\begin{aligned}
 & \sum_{n_1, n_2 \in \mathbb{Z}} \frac{\partial^2 \tilde{F}_0}{\partial t_{n_1} \partial t_{n_2}} \frac{1}{\sqrt{2\pi}^2} \int_0^\infty \int_0^\infty z_1^{n_1+1} z_2^{n_2+1} e^{-y_1 z_1/2 - y_2 z_2/2} dz_1 dz_2 \\
 (50) \quad &= \sum_{n \geq 0} (2n+1) \left(\frac{y_1^{n-1/2}}{y_2^{n+3/2}} + \frac{y_2^{n-1/2}}{y_1^{n+3/2}} \right) \\
 &+ \sum_{n_1, n_2 \geq 0} \frac{\partial^2 F_0}{\partial t_{n_1} \partial t_{n_2}}(\mathbf{t}) \cdot (2n_1+1)!! y_1^{-n_1-3/2} \cdot (2n_2+1)!! y_2^{-n_2-3/2}.
 \end{aligned}$$

By restricting to the small phase space and taking the summations,

$$\begin{aligned}
 & \sum_{n_1, n_2 \in \mathbb{Z}} \frac{\partial^2 \tilde{F}_0}{\partial t_{n_1} \partial t_{n_2}}(t_0) \int_0^\infty \int_0^\infty z_1^{n_1+1} z_2^{n_2+1} e^{-y_1 z_1/2 - y_2 z_2/2} \frac{dz_1 dz_2}{\sqrt{2\pi}^2} \\
 (51) \quad &= i_{y_1, y_2} \frac{y_1 + y_2}{y_1^{1/2} y_2^{1/2} (y_1 - y_2)^2} + i_{y_2, y_1} \frac{y_1 + y_2}{y_1^{1/2} y_2^{1/2} (y_1 - y_2)^2} \\
 &+ \left(\frac{y_1 + y_2 - 4t_0}{(y_1 - 2t_0)^{1/2} (y_2 - 2t_0)^{1/2} (y_1 - y_2)^2} - \frac{y_1 + y_2}{y_1^{1/2} y_2^{1/2} (y_1 - y_2)^2} \right).
 \end{aligned}$$

The second line on the right-hand side is understood as a Taylor series in t_0 . If one takes furthermore $t_0 = 0$,

$$\begin{aligned}
 & \sum_{n_1, n_2 \in \mathbb{Z}} \frac{\partial^2 \tilde{F}_0}{\partial t_{n_1} \partial t_{n_2}} \Big|_{t_k=0} \int_0^\infty \int_0^\infty z_1^{n_1+1} z_2^{n_2+1} e^{-y_1 z_1/2 - y_2 z_2/2} \frac{dz_1 dz_2}{\sqrt{2\pi}^2} \\
 (52) \quad &= i_{y_1, y_2} \frac{y_1 + y_2}{y_1^{1/2} y_2^{1/2} (y_1 - y_2)^2} + i_{y_2, y_1} \frac{y_1 + y_2}{y_1^{1/2} y_2^{1/2} (y_1 - y_2)^2}.
 \end{aligned}$$

This gives an interpretation of [28, (26)] by ghost variables.

2.7. Emergence of a conformal field theory in quantum deformation theory. Now we consider the problem of including free energy of higher genera. In the quantum deformation theory approach in [30], we consider the special deformation of the Airy curve $y = x^2$ given by (44) and quantize it to the field:

$$(53) \quad \hat{x}(y) := \sum_{n \geq 0} y^{n-1/2} \frac{t_n - \delta_{n,1}}{(2n-1)!!} \cdot + \sum_{n \geq 0} y^{-n-3/2} (2n+1)!! \frac{\partial}{\partial t_n},$$

and define a regularized product $\hat{x}(y)^{\odot 2} = \hat{x}(y) \odot \hat{x}(y)$. Furthermore, the Witten-Kontsevich tau-function is shown to be uniquely determined by:

$$(54) \quad \hat{x}(y)^{\odot 2} Z_{WK} = 0,$$

this condition is shown to be equivalent to the DVV Virasoro constraints. A conformal field whose Fock space living near the infinity of the Airy curve emerge in this framework. In fact, we understand $\hat{x}(y)$ as a field of operators on the Airy curve, with y as a local coordinate near the infinity of the Airy curve. We understand Z_{WK} as an element of the 2-reduced bosonic Fock space consisting of symmetric functions generated by the odd power functions p_{2n+1} 's, associated with the infinity of the Airy curve.

2.8. All genera one-point function on the t_0 -line. Define the all-genera one-point function restricted to the t_0 -line by:

$$(55) \quad \tilde{t}_0(y; \lambda) = -y^{1/2} + \frac{t_0}{y^{1/2}} + \sum_{n \geq 0} \frac{(2n+1)!!}{y^{n+3/2}} \sum_{g \geq 0} \lambda^{2g} \frac{\partial F_g}{\partial t_n}(t_0).$$

Note we have:

$$\frac{\partial F_g}{\partial t_n}(t_0) = \sum_{m \geq 0} \langle \tau_n \tau_0^m \rangle_g \frac{t_0^m}{m!}$$

where the following selection rule has to be satisfied:

$$(56) \quad n = 3g - 2 + m$$

When $g = 0$, $n = m - 2$, we have:

$$\sum_{m \geq 2} \frac{(2m-3)!!}{y^{m-1/2}} \langle \tau_{m-2} \tau_0^m \rangle_0 \frac{t_0^m}{m!} = \sum_{m \geq 2} \frac{(2m-3)!!}{y^{m-1/2}} \frac{t_0^m}{m!}.$$

For $g \geq 1$, we have

$$\begin{aligned} & \sum_{m \geq 0} \frac{(6g-3+2m)!!}{y^{3g+m-1/2}} \langle \tau_{3g-2+m} \tau_0^m \rangle_g \frac{t_0^m}{m!} \\ &= \frac{1}{24^g g!} \sum_{m \geq 0} \frac{(6g-3+2m)!!}{y^{3g+m-1/2}} \frac{t_0^m}{m!} \\ &= \frac{(6g-3)!!}{24^g g! y^{3g-1/2}} \left(1 - \frac{2t_0}{y}\right)^{-3g+1/2} \\ &= \frac{(6g-3)!!}{24^g g! (y - 2t_0)^{3g-1/2}} \end{aligned}$$

where we have used the following fact due to Witten [25]:

$$(57) \quad \langle \tau_{3g-2} \rangle_g = \frac{1}{24^g g!}$$

and the puncture equation. So we get:

$$(58) \quad p(y, t_0; \lambda) = \sum_{g \geq 0} \frac{(6g-3)!! \lambda^{2g}}{24^g g! (y-2t_0)^{3g-1/2}}.$$

After taking derivative in t_0 , one gets:

$$(59) \quad \partial_{t_0} p(y, t_0; \lambda) = \sum_{g \geq 0} \frac{(6g-1)!! \lambda^{2g}}{24^g g! (y-2t_0)^{3g-1/2}}.$$

In particular

$$\partial_{t_0} p(y, t_0 = 0; \lambda) = \sum_{g \geq 0} \frac{\lambda^{2g}}{24^g g!} \frac{(6g-1)!!}{y^{3g+1/2}}.$$

This series is closely related to the Airy functions. Compare it with (2). See also §6.10.

2.9. Quantum corrections to the versal deformation of the Airy curve. More explicitly, one has:

$$\begin{aligned} p &= -(y-2t_0)^{1/2} + \frac{\lambda^2}{8}(y-2t_0)^{-5/2} + \frac{105\lambda^4}{128}(y-2t_0)^{-11/2} \\ &+ \frac{25025\lambda^6}{1024}(y-2t_0)^{-17/2} + \frac{56581525\lambda^8}{32768}(y-2t_0)^{-23/2} \\ &+ \frac{58561878375\lambda^{10}}{262144}(y-2t_0)^{-29/2} + \dots \end{aligned}$$

By Lagrangian inversion:

$$\begin{aligned} (y-2t_0)^{1/2} &= p \left(1 + \frac{\lambda^2}{8} p^{-6} + \frac{95\lambda^4}{128} p^{-12} + \frac{23425}{\lambda^6} 1024 p^{-18} \right. \\ &\quad \left. + \frac{54230715\lambda^8}{32768} p^{-24} + \frac{56875278215\lambda^{10}}{262144} p^{-30} + \dots \right), \end{aligned}$$

and so

$$\begin{aligned} y-2t_0 &= p^2 \left(1 + \frac{\lambda^2}{4} p^{-6} + \frac{3\lambda^4}{2} p^{-12} + \frac{735\lambda^6}{16} p^{-18} \right. \\ &\quad \left. + \frac{13265\lambda^8}{4} p^{-24} + \frac{27799785\lambda^{10}}{64} p^{-30} + \dots \right). \end{aligned}$$

This once again indicates that one should study deformations more general than those in traditional deformation theory.

2.10. All genera two-point function on the t_0 -line.

2.11. The n -point functions in genus 0. In [30, Section 4.4], we have shown that the special deformation of the Airy curve gives rise a series

$$(60) \quad y = x^2 + \cdots,$$

where \cdots consists of terms in positive powers of t_0, t_1, \dots . Let $L = y^{1/2}$ and write it as Laurent series. In [30] we define the one-point function in genus zero on the big phase space by:

$$(61) \quad \langle\langle\phi_j\rangle\rangle_0 = \frac{1}{(2j+3)!!} \operatorname{res}(L^{2j+3}dx),$$

and define n -point function ($n \geq 2$) by taking derivatives:

$$(62) \quad \langle\langle\phi_{j_1}, \dots, \phi_{j_n}\rangle\rangle_0 = \frac{\partial}{\partial t_{j_1}} \langle\langle\phi_{j_2} \cdots \phi_{j_n}\rangle\rangle_0.$$

The following result [30, Theorem 6.4] tells us how to recover the genus zero free energy of the 2D topological gravity from the special deformation of the Airy curve:

$$(63) \quad \langle\langle\phi_{j_1}, \dots, \phi_{j_n}\rangle\rangle_0 = \frac{\partial^n F_0}{\partial t_{j_1} \cdots \partial t_{j_n}}.$$

3. EMERGENCE OF SATO GRASSMANNIAN AND TAU-FUNCTION

Our discussion of the special deformation theory leads naturally to Sato's Grassmannian and tau-function of KP hierarchy. In this Section we will review this beautiful theory. In the process we will introduce normalized basis following [4, 5] and formulate Theorem 3.1 expressing Sato tau-function in terms of a Bogoliubov transform.

3.1. Sato's semi-infinite Grassmannian. Restrict to the small phase space, i.e., take all $t_n = 0$ except for t_0 . Then one gets a sequence of series of the following form:

$$\begin{aligned} L^{2n+1} &= (x^2 + 2t_0)^{(2n+1)/2} = x^{2n+1} \left(1 + \frac{2t_0}{x^2}\right)^{(2n+1)/2} \\ &= x^{2n+1} + (2n+1)t_0 x^{2n-1} + \frac{(2n+1)(2n-1)}{2} t_0^2 x^{2n-3} + \cdots. \end{aligned}$$

Such a sequence determines a point in Sato's Grassmannian. Let H be the space consisting of the formal Laurent series $\sum_{n \in \mathbb{Z}} a_n z^{n-1/2}$, such that $a_n = 0$ for $n \gg 0$. Take $\{z^{n-1/2}\}_{n \in \mathbb{Z}}$ as a semi-infinite basis. This means an element $\sum_{n \in \mathbb{Z}} a_n z^{n-1/2} \in H$ has as coordinates a sequence

of numbers $\{a_n\}_{n \in \mathbb{Z}}$, semi-infinite in the sense that $a_n = 0$ for $n \gg 0$. This space has a natural inner product and a symplectic structure:

$$(64) \quad \langle f, g \rangle := \text{res}(f(z)g(z)dz),$$

$$(65) \quad \omega(f, g) := \text{res}(f(-z)g(z)dz).$$

It is clear that one has a decomposition:

$$(66) \quad H = H_+ \oplus H_-,$$

where $H_+ = \{\sum_{n \geq 1} a_n z^{n-1/2} \in H\}$, $H_- = \{\sum_{n \leq 0} a_n z^{n-1/2} \in H\}$. Denote by $\pi_{\pm} : H \rightarrow H_{\pm}$ the natural projections. The big cell of Sato Grassmannian $\text{Gr}_{(0)}$ consists of linear subspaces $U \subset H$ such that $\pi_+|_U : U \rightarrow H_+$ is an isomorphism.

3.2. Admissible basis and Plücker coordinates. Suppose that $U \in \text{Gr}_{(0)}$. Since $\{z^{n+1/2}\}_{n \geq 0}$ is a basis of H_+ , elements of the form

$$(67) \quad f_n(z) := \pi_+^{-1}(z^{n+1/2}) = z^{n+1/2} + \sum_{k < n} c_{n,k} z^{k+1/2}$$

form a basis of U . This shows a subspace U in the big cell of Sato Grassmannian has a basis of the form $\{f_n(z) = z^{n+1/2} + \dots\}_{n \geq 0}$, where \dots denote lower order terms. Conversely, a linear subspace of H with a basis of this form lies in $\text{Gr}_{(0)}$. Such a basis will be called an admissible basis.

Given an admissible basis $\{f_n = z^{n+1/2} + \sum_{k < n} c_{n,k} z^{k+1/2}\}_{n \geq 0}$, for any $N \geq 1$, consider the expansion of the wedge product:

$$f_0 \wedge f_1 \wedge \dots \wedge f_{N-1}.$$

First of all we have a term of the form:

$$z^{1/2} \wedge z^{3/2} \wedge \dots \wedge z^{N-1/2},$$

all the other terms are of the form:

$$\begin{aligned} & z^{-m_1-1/2} \wedge \dots \wedge z^{-m_{l-1}-1/2} \wedge z^{-m_l-1/2} \\ \wedge & \quad z^{1/2} \wedge \dots \wedge \widehat{z^{n_l+1/2}} \wedge \dots \wedge \widehat{z^{n_1+1/2}} \wedge \dots \wedge z^{N-1/2}, \end{aligned}$$

for some nonnegative numbers

$$(68) \quad m_1 > i_2 > \dots > m_l \geq 0, \quad n_1 > j_2 > \dots > n_l \geq 0.$$

It is straightforward to see that the coefficient of this term can be found as follows: Use the coefficients of f_n to form the following matrix:

$$\begin{pmatrix} \cdots & c_{0,-3} & c_{0,-2} & c_{0,-1} & 1 & & \\ \cdots & c_{1,-3} & c_{1,-2} & c_{1,-1} & c_{1,0} & 1 & \\ \cdots & c_{2,-3} & c_{2,-2} & c_{2,-1} & c_{2,0} & c_{2,1} & 1 \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & c_{N-1,-3} & c_{N-1,-2} & c_{N-1,-1} & c_{N-1,0} & c_{N-1,1} & \cdots & 1 \end{pmatrix}$$

from this matrix one can form an $N \times N$ -matrix by taking among the columns index by negative integers those indexed by $-m_1 - 1, -m_2 - 1, \dots, -m_l - 1$, and delete among the columns indexed by nonnegative integers that indexed by n_l, n_{l-1}, \dots, n_1 . Denote by $C_{(m_1, \dots, m_l | n_1, \dots, n_l)}$ this matrix. It is easy to see that the determinant of this matrix is independent of N . In fact, the determinant of this matrix is equal to the determinant of a matrix $B_{(m_1, \dots, m_l | n_1, \dots, n_l)}$ of size $(n_1 + 1) \times (n_1 + 1)$ by removing from $C_{(m_1, \dots, m_l | n_1, \dots, n_l)}$ the columns indexed by $n > n_1$ and the corresponding rows. For example, $B_{(0|0)} = (c_{0,-1})$, $B_{(1|0)} = (c_{0,-2})$, $B_{(0|1)} = \begin{pmatrix} c_{0,-1} & 1 \\ c_{1,-1} & c_{1,0} \end{pmatrix}$. For comparison, note $B_{(100,1|2,1)}$ is of the form:

$$\begin{pmatrix} c_{0,-101} & c_{0,-2} & 1 \\ c_{1,-101} & c_{1,-2} & c_{1,0} \\ c_{2,-101} & c_{2,-2} & c_{2,0} \end{pmatrix},$$

it is of size 3×3 . On the other hand, $B_{(2,1|100,1)}$ is of the form:

$$\begin{pmatrix} c_{0,-3} & c_{0,-2} & 1 & & & \\ c_{1,-3} & c_{1,-2} & c_{1,0} & 0 & & \\ c_{2,-3} & c_{2,-2} & c_{2,0} & 1 & & \\ c_{3,-3} & c_{3,-2} & c_{3,0} & c_{3,2} & 1 & \\ \vdots & \vdots & \vdots & \vdots & & \\ c_{99,-3} & c_{99,-2} & c_{99,0} & c_{99,2} & \cdots & 1 \\ c_{100,-3} & c_{100,-2} & c_{100,0} & c_{100,2} & \cdots & c_{100,99} \end{pmatrix}$$

It is of size 101×101 .

Denote by $v_{(m_1, \dots, m_l | n_1, \dots, n_l)}$ the following expression:

$$z^{-m_1-1/2} \wedge \cdots \wedge z^{-m_{l-1}-1/2} \wedge z^{-j_l-1/2} \\ \wedge z^{1/2} \wedge \cdots \wedge \widehat{z^{n_l+1/2}} \wedge \cdots \wedge \widehat{z^{n_1+1/2}} \wedge \cdots,$$

then one has

$$(69) \quad f_0 \wedge f_1 \wedge \cdots = \sum \det B_{(m_1, \dots, m_l | n_1, \dots, n_l)} \cdot v_{(m_1, \dots, m_l | n_1, \dots, n_l)}.$$

See [13] for a similar formula.

3.3. Normalized basis and Plücker coordinates. In computing the determinant of the matrix $B_{(m_1, \dots, m_l | n_1, \dots, n_l)}$, one can use Gauss elimination to perform some row operations. There are $n_1 + 1 - l$ places where the entries are 1, so the result will be

$$(70) \quad \det B_{(m_1, \dots, m_l | n_1, \dots, n_l)} = \pm \det A_{(m_1, \dots, m_l | n_1, \dots, n_l)}$$

for some matrix $A_{(m_1, \dots, m_l | n_1, \dots, n_l)}$ of size $l \times l$.

This can be done more geometrically as follows. By Gauss elimination one can see that every $U \in \text{Gr}_{(0)}$ has an admissible basis of the form

$$(71) \quad f_n = z^{n+1/2} + \sum_{m \geq 0} a_{n,m} z^{-m-1/2},$$

such a basis will be called a normalized basis. The coefficients $\{a_{n,m}\}$ are called the affine coordinates on the big cell [4]. Then one has

$$\begin{aligned} & f_1 \wedge f_2 \wedge \dots \\ &= \sum \alpha_{m_1, \dots, m_l; n_1, \dots, n_l} \cdot z^{-m_1-1/2} \wedge \dots \wedge z^{-m_l-1/2} \\ & \quad \wedge z^{1/2} \wedge \dots \wedge \widehat{z^{n_l+1/2}} \wedge \dots \wedge \widehat{z^{n_1+1/2}} \wedge \dots, \end{aligned}$$

where $m_1 > m_2 > \dots > m_l \geq 0$, $n_1 > n_2 > \dots > n_l \geq 0$ are two sequences of integers, and

$$(72) \quad \alpha_{m_1, \dots, m_l; n_1, \dots, n_l} = (-1)^{n_1 + \dots + n_l} \begin{vmatrix} a_{n_1, m_1} & \dots & a_{n_1, m_l} \\ \vdots & & \vdots \\ a_{n_l, m_1} & \dots & a_{n_l, m_l} \end{vmatrix}$$

3.4. The fermionic Fock space. It is well-known that the notation $(m_1, \dots, m_l | n_1, \dots, n_l)$ is the Frobenius notation of a partition $\mu = (\mu_1, \dots, \mu_l)$, and numbers m_i and n_i are given by:

$$(73) \quad m_i = \mu_i - i, \quad n_i = \mu_i^t - i,$$

where μ^t is the conjugate partition of μ . Furthermore, one has

$$\begin{aligned} & z^{-m_1-1/2} \wedge \dots \wedge z^{-m_{l-1}-1/2} \wedge z^{-j_l-1/2} \\ & \wedge z^{1/2} \wedge \dots \wedge \widehat{z^{n_l+1/2}} \wedge \dots \wedge \widehat{z^{n_1+1/2}} \wedge \dots \\ &= z^{1/2-\mu_1} \wedge z^{3/2-\mu_2} \wedge \dots. \end{aligned}$$

The above discussions naturally lead one to fermionic Fock space. For a sequence $\mathbf{a} = (a_1, a_2, \dots)$ of half-integers such that $a_1 < a_2 < \dots$. The elements in the set $\{a_i \mid a_i < 0\}$ will be called bubbles of \mathbf{a} , the elements in the set $(\mathbb{Z}_{\geq 0} + \frac{1}{2}) - \{a_1, a_2, \dots\}$ will be called holes of \mathbf{a} . It is

clear that \mathbf{a} can have only finitely many bubbles. We say \mathbf{a} is admissible if it also has only finitely many holes. For an admissible sequence \mathbf{a} , let

$$(74) \quad |\mathbf{a}\rangle := z^{a_1} \wedge z^{a_2} \wedge \cdots \in \Lambda^{\frac{\infty}{2}}(H).$$

Suppose that \mathbf{a} has k bubbles and l holes, Following Dirac, $|\mathbf{a}\rangle$ describes a state with k electrons and l positrons, so one defines the charge of \mathbf{a} to be $l - k$. The fermionic Fock space \mathcal{F} is the space of expressions of form:

$$(75) \quad \sum_{\mathbf{a}} c_{\mathbf{a}} |\mathbf{a}\rangle,$$

where the sum is taken over admissible sequences. One has a decomposition:

$$(76) \quad \mathcal{F} = \sum_{n \in \mathbb{Z}} \mathcal{F}^{(n)},$$

where $\mathcal{F}^{(n)}$ are generated by $|\mathbf{a}\rangle$ with charge n . The space $\mathcal{F}^{(0)}$ is particularly interesting. Its basis vectors correspond to partitions $\mu = (\mu_1, \mu_2, \mu_3, \dots)$, where $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_l > \mu_{l+1} = \mu_{l+2} = \cdots = 0$:

$$(77) \quad |\mu\rangle := z^{-(\mu_1-1/2)} \wedge z^{-(\mu_2-3/2)} \wedge \cdots.$$

When all $\mu_i = 0$, the partition is called the empty partition, the corresponding vector is called the fermionic vacuum vector and is denoted by $|0\rangle$:

$$(78) \quad |0\rangle := z^{1/2} \wedge z^{3/2} \wedge \cdots.$$

For $n \in \mathbb{Z}$, define

$$(79) \quad |n\rangle := z^{n+1/2} \wedge z^{n+3/2} \wedge \cdots.$$

This is a vector in $\mathcal{F}^{(n)}$.

3.5. Creators and annihilators on \mathcal{F} . As in the case of ordinary Grassmann algebra, one can consider exterior products and inner products. For $r \in \mathbb{Z} + \frac{1}{2}$, define operator $\psi_r : \Lambda^{\frac{\infty}{2}}(H) \rightarrow \Lambda^{\frac{\infty}{2}}(H)$ by

$$(80) \quad \psi_r |\mathbf{a}\rangle = z^r \wedge |\mathbf{a}\rangle,$$

and let $\psi_r^* : \Lambda^{\frac{\infty}{2}}(H) \rightarrow \Lambda^{\frac{\infty}{2}}(H)$ be defined by:

$$(81) \quad \psi_r^* |\mathbf{a}\rangle = \begin{cases} (-1)^{k+1} \cdot z^{a_1} \wedge \cdots \wedge \widehat{z^{a_k}} \wedge \cdots, & \text{if } a_k = -r \text{ for some } k, \\ 0, & \text{otherwise.} \end{cases}$$

These operators have charge -1 and 1 respectively.

The anti-commutation relations for these operators are

$$(82) \quad [\psi_r, \psi_s^*]_+ := \psi_r \psi_s^* + \psi_s^* \psi_r = \delta_{-r,s} id$$

and other anti-commutation relations are zero. It is clear that for $r > 0$,

$$(83) \quad \psi_r|0\rangle = 0, \quad \psi_r^*|0\rangle = 0,$$

so the operators $\{\psi_r, \psi_r^*\}_{r>0}$ are called the fermionic annihilators. For a partition μ , let $z^{-m_1-1/2}, \dots, z^{-m_k-1/2}$ be the positions of the electrons, $z^{n_1+1/2}, \dots, z^{n_k+1/2}$ be the positions of the positrons, $m_1 > m_2 > \dots > m_k \geq 0$, $n_1 > \dots > n_k \geq 0$, then $(m_1, m_2, \dots, m_k | n_1, n_2, \dots, n_k)$ is the Frobenius notation for μ , and one has

$$(84) \quad |\mu\rangle = (-1)^{n_1+n_2+\dots+n_k} \psi_{-m_1-\frac{1}{2}} \psi_{-n_1-\frac{1}{2}}^* \cdots \psi_{-m_k-1/2} \psi_{-n_k-1/2}^* |0\rangle,$$

and so the operators $\{\psi_{-r}, \psi_{-r}^*\}_{r>0}$ are the fermionic creators.

3.6. Elements of $\mathcal{F}^{(0)}$ associated with points on big cell of Sato Grassmannian. Now we come to formulate the main result of this Section.

Theorem 3.1. *Suppose that U is given by a normalized basis*

$$\{f_n = z^{n+1/2} + \sum_{m \geq 0} a_{n,m} z^{-m-1/2}\},$$

the one has

$$(85) \quad |U\rangle = e^A |0\rangle,$$

where $A : \mathcal{F}^{(0)} \rightarrow \mathcal{F}^{(0)}$ is a linear operator

$$(86) \quad A = \sum_{m,n \geq 0} a_{n,m} \psi_{-m-1/2} \psi_{-n-1/2}^*.$$

This is of course just a reformulation of §3.3.

3.7. The boson-fermion correspondence. In the rest of this Section, we will recall boson-fermion correspondence and its applications to Sato tau-function. The materials are well-known and can be found in e.g. [19]. They are included here to fix the notations for later Sections. For any integer n , define an operator α_n on the fermionic Fock space \mathcal{F} as follows:

$$\alpha_n = \sum_{r \in \mathbb{Z} + \frac{1}{2}} : \psi_{-r} \psi_{r+n}^* :$$

When $n \neq 0$, the effect of α_n on the vector $|\mathbf{a}\rangle$ is the same as the action of the shift operator $s_n : H \rightarrow H$ on $\Lambda^{\frac{\infty}{2}}(H)$. When $n = 0$, α_0 is the charge operator or fermionic number operator:

$$(87) \quad \alpha_0 = \sum_{r>0} (\psi_{-r} \psi_r^* - \psi_{-r}^* \psi_r).$$

They satisfy the following commutation relations:

$$(88) \quad [\alpha_m, \alpha_n] = m\delta_{m,-n}.$$

These operators can also arise in the following way. Define the fermionic generating function:

$$(89) \quad \psi(\xi) = \sum_{r \in \mathbb{Z}+1/2} \psi_r \xi^{-r-1/2}, \quad \psi^*(\xi) = \sum_{r \in \mathbb{Z}+1/2} \psi_r^* \xi^{-r-1/2}.$$

The commutation relations (82) is equivalent to the following operator product expansion:

$$(90) \quad \psi(\xi)\psi^*(\eta) =: \psi(\xi)\psi^*(\eta) : + \frac{1}{\xi - \eta},$$

$$(91) \quad \psi(\xi)\psi^*(\eta) =: \psi(\xi)\psi(\eta) :,$$

$$(92) \quad \psi^*(\xi)\psi^*(\eta) =: \psi^*(\xi)\psi^*(\eta) :.$$

Define the generating function of the operators α_n by

$$(93) \quad \alpha(\xi) := \sum_{n \in \mathbb{Z}} \alpha_n \xi^{-n-1}.$$

The fields of operators $\alpha(\xi)$, $\psi(\xi)$ and $\psi^*(\xi)$ are related as follows:

$$(94) \quad \alpha(\xi) =: \psi(\xi)\psi^*(\xi) :.$$

The commutation relations (88) is equivalent to the following OPE:

$$(95) \quad \alpha(\xi)\alpha(\eta) =: \alpha(\xi)\alpha(\eta) : + \frac{1}{(\xi - \eta)^2}.$$

One also has the following OPE's:

$$(96) \quad \alpha(\xi)\psi(\eta) = \frac{\psi(\xi)}{\xi - \eta} +: \psi(\xi)\psi^*(\xi)\psi(\eta) :,$$

$$(97) \quad \alpha(\xi)\psi^*(\eta) = -\frac{\psi^*(\xi)}{\xi - \eta} +: \psi(\xi)\psi^*(\xi)\psi^*(\eta) :.$$

There are equivalent to the following commutation relations:

$$(98) \quad [\alpha_m, \xi_r] = \psi_{m+r},$$

$$(99) \quad [\alpha_m, \xi_r^*] = -\psi_{m+r}^*.$$

Consider the space of symmetric functions:

$$(100) \quad \Lambda = \sum_{n \geq 0}^{\infty} \Lambda_n,$$

where Λ_n is the space of homogeneous symmetric functions of degree n . Let $\mathcal{B} = \Lambda[w, w^{-1}]$ be the bosonic Fock space, where w is a formal variable. Then the boson-fermion correspondence is a linear isomorphism $\Phi : \mathcal{F} \rightarrow \mathcal{B}$ given by

$$(101) \quad |\mathbf{a}\rangle \mapsto w^m \langle \underline{0}_m | e^{\sum_{n=1}^{\infty} \frac{pn}{n} \alpha_n} |\mathbf{a}\rangle, \quad |\mathbf{a}\rangle \in \mathcal{F}^{(m)}$$

where $|\underline{0}_m\rangle = z^{\frac{1}{2}+m} \wedge z^{\frac{3}{2}+m} \wedge \dots$. Restricting to $\mathcal{F}^{(0)}$, Φ induces an isomorphism between $\mathcal{F}^{(0)}$ and Λ . Explicitly, this isomorphism is given by [19, Theorem 9.4]:

$$(102) \quad s_\mu = \langle 0 | e^{\sum_{n=1}^{\infty} \frac{pn}{n} \alpha_n} | \mu \rangle.$$

Now combining (117), (114), (120) and (102), one can get (123).

3.8. Action of fermionic operators on bosonic Fock space under boson-fermion correspondence. It is very interesting to understand how the actions of the operators α_n , ψ_r and ψ_r^* originally on the fermionic Fock space get transformed to actions on the bosonic Fock space after the boson-fermion correspondence. The following are well-known (see e.g. [19]):

$$(103) \quad \Phi(\alpha_n |\mathbf{a}\rangle) = \begin{cases} n \frac{\partial}{\partial p_n} \Phi(|\mathbf{a}\rangle), & \text{if } n > 0, \\ -p_n \cdot \Phi(|\mathbf{a}\rangle), & \text{if } n < 0, \end{cases}$$

Hence

$$(104) \quad \Phi(\alpha(\xi) |\mathbf{a}\rangle) = \left(\sum_{n \geq 1} n \frac{\partial}{\partial p_n} \xi^{-n-1} + \sum_{n=1}^{\infty} \xi^{n-1} p_n \cdot \right) (\Phi(|\mathbf{a}\rangle)).$$

For the fermionic operators, one needs to introduce the vertex operators:

$$(105) \quad \Psi(\xi) =: e^{\varphi(\xi)} :, \quad \Psi^*(\xi) = e^{-\varphi(\xi):},$$

where the field φ is defined by

$$(106) \quad \varphi(\xi) = \sum_{n \in \mathbb{Z} - \{0\}} \frac{\alpha_n}{-n} \xi^{-n} + \alpha_0 \log \xi + K,$$

which is an integral of the field $\alpha(\xi)$. The integration constant K is not well-defined as operator, but it will be treated as a creator, and formally one requires the following commutation rule:

$$(107) \quad [K, \alpha_n] = \delta_{n,0}.$$

Then the vertex operators are given by

$$(108) \quad \Psi(\xi) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n} \xi^n\right) \exp\left(-\sum_{n=1}^{\infty} \xi^{-n} \frac{\partial}{\partial p_n}\right) e^K \xi^{\alpha_0},$$

$$(109) \quad \Psi^*(\xi) = \exp\left(-\sum_{n=1}^{\infty} \frac{p_n}{n} \xi^n\right) \exp\left(\sum_{n=1}^{\infty} \xi^{-n} \frac{\partial}{\partial p_n}\right) e^{-K} \xi^{-\alpha_0},$$

where the actions of e^K and ξ^{α_0} are defined by:

$$(110) \quad (e^K f)(z, \mathbf{T}) = z \cdot f(z, \mathbf{T}), \quad (\xi^{\alpha_0} f)(z, \mathbf{T}) = f(\xi z, \mathbf{T}).$$

Under the boson-fermion correspondence,

$$(111) \quad \Phi(\psi(\xi)|\mathbf{a}) = \Psi(\xi)\Phi(|\mathbf{a})),$$

$$(112) \quad \Phi(\psi^*(\xi)|\mathbf{a}) = \Psi^*(\xi)\Phi(|\mathbf{a})),$$

3.9. Sato's construction of tau-functions. Define a metric on \mathcal{F} such that $\{|\mathbf{a}\rangle \mid \mathbf{a} \text{ is admissible}\}$ is an orthonormal basis. The fermionic vacuum is define by:

$$(113) \quad |0\rangle := z^{1/2} \wedge z^{3/2} \wedge \dots$$

Following notations in physics literature the inner product of a vector $|v\rangle$ with $|0\rangle$ will be denoted by $\langle 0|v\rangle$. One can easily see that

$$(114) \quad \begin{aligned} & \dots \wedge z^{-3/2} \wedge z^{-1/2} \wedge |\mathbf{a}\rangle \\ &= \langle 0|\mathbf{a}\rangle \cdot (\dots \wedge z^{-3/2} \wedge z^{-1/2} \wedge z^{1/2} \wedge z^{3/2} \wedge \dots). \end{aligned}$$

Let $s_m : H \rightarrow H$ be the shift operator defined by:

$$(115) \quad s_m(z^{n-1/2}) = z^m \cdot z^{n-1/2} = z^{n+m-1/2},$$

and let $\Gamma_+(\mathbf{T})$ be defined by

$$(116) \quad \Gamma_+(\mathbf{T}) = \exp \sum_{n=1}^{\infty} T_n s_n$$

Sato associated a tau-function τ_U to a subspace U spanned by an admissible basis of the form $\{f_n(z) = z^{n+1/2} + \sum_{j < n} a_{n,j} z^{j+1/2}\}_{n \geq 0}$ as follows:

$$(117) \quad \begin{aligned} & \tau_U(\mathbf{T}) \cdot (\dots \wedge z^{-3/2} \wedge z^{-1/2} \wedge z^{1/2} \wedge z^{3/2} \wedge \dots) \\ &= \dots \wedge z^{-3/2} \wedge z^{-1/2} \wedge \Gamma_+(\mathbf{T})(f_0(z)) \wedge \Gamma_+(\mathbf{T})(f_1(z)) \wedge \dots \end{aligned}$$

Combining (98) with (115) and combining (114) with (117), one gets:

$$(118) \quad \tau_U(\mathbf{T}) = \langle 0|e^{\sum_{n \geq 1} T_n \alpha_n}|U\rangle,$$

where $|U\rangle \in \mathcal{F}^{(0)}$ is defined by:

$$(119) \quad |U\rangle := f_0(z) \wedge f_1(z) \wedge \dots$$

Since $\{|\mu\rangle\}$ form a basis of $\mathcal{F}^{(0)}$, there exists $c_\mu(U)$ such that

$$(120) \quad |U\rangle = \sum_{\mu} c_\mu(U) \cdot |\mu\rangle.$$

The coefficients c_μ can be found using Theorem 3.1. One can understand $\tau_U(\mathbf{T})$ as the inner product of $|U\rangle$ with $e^{\sum_{n \geq 1} T_n \alpha_{-n}} |0\rangle$. By boson-fermion correspondence (102),

$$(121) \quad e^{\sum_{n \geq 1} T_n \alpha_{-n}} |0\rangle = \sum_{\mu} s_\mu(\mathbf{T}) |\mu\rangle,$$

where $s_\mu(\mathbf{T})$ are the Schur functions defined as follows [18]:

$$(122) \quad s_\mu = \sum_{\nu} \frac{\chi_{\nu}^{\mu}}{z_{\nu}} p_{\nu}, \quad p_{\nu} = \prod_i p_{\nu_i}, \quad p_n = n T_n.$$

Therefore the tau-function admits an expansion:

$$(123) \quad \tau_U(\mathbf{T}) = \sum_{\mu} c_\mu(U) s_\mu(\mathbf{T}),$$

where the sum is taken over all partitions.

In the above the tau-function is constructed as a formal power series. See Segal-Wilson [23] for an analytic construction.

Sato's construction of the tau-function establishes a connection between the theory of integrable hierarchies with conformal field theory:

$$\begin{aligned} \text{Sato Grassmannian} &\rightarrow \text{Fermionic Fock Space} \rightarrow \text{Bosonic Fock Space} \\ U \in \text{Gr}_{(0)} &\rightarrow |U\rangle \in \mathcal{F}^{(0)} \rightarrow \tau_U \in \Lambda \end{aligned}$$

He understood the theory of the integrable hierarchies as a dynamical systems on the infinite-dimensional Grassmannian. One should reverse the arrows in the above picture to get:

$$\tau_U = \langle 0 | e^{\sum_{n \geq 1} \frac{p_n}{n} \alpha_n} |U\rangle \in \Lambda \rightarrow e^{\sum_{n \geq 1} \frac{p_n}{n} \alpha_n} |U\rangle \in \mathcal{F}^{(0)} \rightarrow e^{\sum_{n \geq 1} \frac{p_n}{n} s_n}(U) \in \text{Gr}_{(0)}.$$

4. BOSONIC AND FERMIONIC N -POINT FUNCTIONS

In this Section we will use the techniques in conformal field theory to understand integrable systems based on Sato's theory.

4.1. Bosonic and fermionic correlation functions. Given $U \in \text{Gr}_{(0)}$, we have seen that it determines a vector $|U\rangle \in \mathcal{F}^{(0)}$. One can define bosonic correlation functions

$$(124) \quad \langle 0 | \alpha(z_1) \cdots \alpha(z_n) |U\rangle$$

and fermionic correlation functions

$$(125) \quad \langle n - m | \psi(z_1) \cdots \psi(z_m) \psi^*(w_1) \cdots \psi^*(w_n) |U\rangle$$

and their mixtures. Next we will show that they naturally appear in the study of integrable systems.

4.2. Hirota bilinear relations. According to Sato [22], $\tau_U(\mathbf{T})$ is a tau-function of the KP hierarchy. Let us recall how this can be shown. First we show that the Hirota bilinear relation is satisfied by $|U\rangle$ (cf. [19, Theorem 9.3]):

$$(126) \quad \sum_{r \in \mathbb{Z} + 1/2} \psi_r^* |U\rangle \otimes \psi_{-r} |U\rangle = 0.$$

With Theorem 3.1, we can give a more straightforward proof as follows. One notes:

$$\begin{aligned} \psi(\xi) |U\rangle &= \sum_{m=0}^{\infty} (\psi_{-m-1/2} \xi^m - \sum_{n \geq 0} A_{n,m} \psi_{-n-1/2} \xi^{-m-1}) |U\rangle, \\ \psi^*(\xi) |U\rangle &= \sum_{a=0}^{\infty} (\psi_{-a-1/2}^* \xi^a + \sum_{b \geq 0} A_{a,b} \psi_{-b-1/2}^* \xi^{-a-1}) |U\rangle, \end{aligned}$$

it follows that

$$\begin{aligned} & \text{res}_{\xi=\infty} (\psi^*(\xi) |U\rangle \otimes \psi(\xi) |U\rangle) \\ &= - \sum_{a \geq 0} \sum_{n \geq 0} A_{n,a} \cdot \psi_{-a-1/2}^* |U\rangle \otimes \psi_{-n-1/2} |U\rangle \\ &+ \sum_{m \geq 0} \sum_{b \geq 0} A_{m,b} \cdot \psi_{-b-1/2}^* |U\rangle \otimes \psi_{-m-1/2} |U\rangle \\ &= 0. \end{aligned}$$

4.3. Wave-function, dual wave-function, and Hirota bilinear relations. The fermionic 1-point functions are also called the wave-function and the dual wave-function respectively:

$$(127) \quad w(\mathbf{T}; \xi) = \langle -1 | e^{\sum_{n \geq 1} \frac{pn}{n} \alpha_n} \psi(\xi) |U\rangle / \tau_U,$$

$$(128) \quad w^*(\mathbf{T}; \xi) = \langle 1 | e^{\sum_{n \geq 1} \frac{pn}{n} \alpha_n} \psi^*(\xi) |U\rangle / \tau_U.$$

By the boson-fermion correspondence,

$$\begin{aligned} (129) \quad w(\mathbf{T}; z) &= \exp\left(\sum_{n=1} T_n z^n\right) \frac{\exp\left(-\sum_{n=1} \frac{z^{-n}}{n} \frac{\partial}{\partial T_n}\right) \tau(\mathbf{T})}{\tau(\mathbf{T})} \\ &= \exp\left(\sum_{n=1} T_n z^n\right) \frac{\tau(\mathbf{T} - [1/z])}{\tau(\mathbf{T})}, \end{aligned}$$

and for the dual wave function:

$$\begin{aligned}
 (130) \quad w^*(\mathbf{T}; z) &= \exp\left(-\sum_{n=1} T_n z^n\right) \frac{\exp\left(\sum \frac{z^{-n}}{n} \frac{\partial}{\partial T_n}\right) \tau(\mathbf{T})}{\tau(\mathbf{T})} \\
 &= \exp\left(\sum_{n=1} T_n z^n\right) \frac{\tau(\mathbf{T} - [1/z])}{\tau(\mathbf{T})}.
 \end{aligned}$$

These are called the Sato formulas. They can be rewritten as follows:

$$(131) \quad w(\mathbf{T}; z) = \frac{X(\mathbf{T}; z) \tau_U(\mathbf{T})}{\tau_U(\mathbf{T})}, \quad w^*(\mathbf{T}; z) = \frac{X^*(\mathbf{T}; z) \tau_U(\mathbf{T})}{\tau_U(\mathbf{T})},$$

where the operators $X(\mathbf{T}; z)$ and $X^*(\mathbf{T}; z)$ defined by

$$(132) \quad X(\mathbf{T}; z) = \exp\left(\sum_{n=1} T_n z^n\right) \cdot \exp\left(-\sum \frac{z^{-n}}{n} \frac{\partial}{\partial T_n}\right),$$

$$(133) \quad X^*(\mathbf{T}; z) = \exp\left(-\sum_{n=1} T_n z^n\right) \cdot \exp\left(\sum \frac{z^{-n}}{n} \frac{\partial}{\partial T_n}\right)$$

are also called the vertex operators. The product of $X^*(\mathbf{T}; w)$ with $X(\mathbf{T}; z)$ is given by:

$$(134) \quad X(\mathbf{T}; z) X^*(\mathbf{T}; w) = \frac{z}{z-w} \cdot X(\mathbf{T}; z, w),$$

where the operator $X(\mathbf{T}; z, w)$ is defined by:

$$(135) \quad X(\mathbf{T}; z, w) = \exp\left(\sum_{n \geq 1} T_n (z^n - w^n)\right) \cdot \exp\left(-\sum_{n \geq 1} \left(\frac{z^{-n}}{n} - \frac{w^{-n}}{n}\right) \frac{\partial}{\partial T_n}\right).$$

By L'Hopital's rule,

$$(136) \quad \lim_{z \rightarrow w} \frac{1}{z-w} (X(\mathbf{T}; z, w) - 1) = \sum_{n \geq 1} n T_n w^{n-1} + \sum_{n \geq 1} w^{-n-1} \frac{\partial}{\partial T_n}.$$

On the fermionic Fock space, this corresponds to

$$(137) \quad \lim_{z \rightarrow w} \left(\psi(z) \psi^*(w) - \frac{1}{z-w} \right) =: \psi(w) \psi^*(w) = \alpha(w).$$

Also by the boson-fermion correspondence, the Hirota bilinear relations (126) becomes:

$$(138) \quad \text{res}_{\xi=\infty} w(\mathbf{x}, \xi) w^*(\mathbf{x}', \xi) = 0.$$

4.4. **Dressing operator and the KP hierarchy.** Note:

$$(139) \quad w(\mathbf{T}; \xi) = \exp\left(\sum_{n=1}^{\infty} T_n \xi^n\right) \cdot \frac{\tau(T_1 - \xi^{-1}, T_2 - \frac{1}{2}\xi^{-2}, \dots)}{\tau(T_1, T_2, \dots)},$$

$$(140) \quad w^*(\mathbf{T}; \xi) = \exp\left(-\sum_{n=1}^{\infty} T_n \xi^n\right) \cdot \frac{\tau(T_1 + \xi^{-1}, T_2 + \frac{1}{2}\xi^{-2}, \dots)}{\tau(T_1, T_2, \dots)}.$$

Write

$$(141) \quad w(\mathbf{T}; \xi) = \exp\left(\sum_{n=1}^{\infty} T_n \xi^n\right) \cdot (1 + \sum_{n=1}^{\infty} w_j \xi^{-n}).$$

The dressing operator M is defined by:

$$(142) \quad M := 1 + \sum_{n=1}^{\infty} w_j \partial_x^{-j}.$$

The dressing operator M and the wave-function w uniquely determine each other:

$$(143) \quad w = M \exp\left(\sum_{n=1}^{\infty} T_n \xi^n\right).$$

Let L be the pseudo-differential operator defined by:

$$(144) \quad L := M \circ \partial_x \circ M^{-1}.$$

It is clear that

$$(145) \quad Lw = \xi \cdot w.$$

From the Hirota bilinear relations one can deduce that

$$(146) \quad \frac{\partial}{\partial T_k} w = (L^k)_+ w.$$

The compatibility condition of the above two equations is

$$(147) \quad \frac{\partial}{\partial T_k} L = [(L^k)_+, L].$$

From this one can also show that:

$$(148) \quad \frac{\partial}{\partial t_k} M = -(L^k)_- M.$$

Indeed, from (147) and (144) one immediately gets:

$$(149) \quad \left[\frac{\partial}{\partial t_k} M \circ M^{-1} + (L^k)_-, L\right] = 0$$

Since $\frac{\partial}{\partial t_k} M \circ M^{-1} + (L^k)_-$ is a pseudodifferential operator with coefficients differential polynomials in a_1, a_2, \dots , so one can see that:

$$(150) \quad \frac{\partial}{\partial t_k} M \circ M^{-1} + (L^k)_- = 0.$$

Here we use the following Lemma easily proves by induction:

Lemma 4.1. *Suppose that $K = \sum_{n=1}^{\infty} b_n \partial_x^{-n}$ and $L = \partial + \sum_{n=1}^{\infty} a_n \partial_x^{-n}$ are pseudo-differential operators such that*

$$(151) \quad [K, L] = 0,$$

then b_n are constants for all $n \geq 1$.

4.5. From wave function to tau-function. Let us give a proof of the following well-known result on the wave function of the KP hierarchy [1] from our point of view:

Proposition 4.2. *Suppose that $w(x; z)$ is the wave-function of KP hierarchy associated to $U \in \text{Gr}_{(0)}$, then*

$$(152) \quad U = \text{span}\{w(0; z), \partial_x w(0; z), \dots\}.$$

Proof. Recall the wave-function is defined by:

$$(153) \quad w(x; z) = \frac{\langle -1 | e^{x\alpha_1} \psi(z) e^{\sum_{m,n \geq 0} A_{m,n} \psi_{-m-1/2} \psi_{-n-1/2}^*} | 0 \rangle}{\langle 0 | e^{x\alpha_1} e^{\sum_{m,n \geq 0} A_{m,n} \psi_{-m-1/2} \psi_{-n-1/2}^*} | 0 \rangle}.$$

We use Leibniz formula to compute its derivatives in x :

$$\begin{aligned} & \partial_x^k w(x; z)|_{x=0} \\ &= \sum_{i=0}^k \binom{k}{i} \partial_x^{k-i} \langle -1 | e^{x\alpha_1} \psi(z) e^{\sum_{m,n \geq 0} A_{m,n} \psi_{-m-1/2} \psi_{-n-1/2}^*} | 0 \rangle \cdot \partial_x^i \frac{1}{\tau(x)} \Big|_{x=0} \\ &= \sum_{i=0}^k \binom{k}{i} \langle -1 | \alpha_1^{k-i} \psi(z) e^{\sum_{m,n \geq 0} A_{m,n} \psi_{-m-1/2} \psi_{-n-1/2}^*} | 0 \rangle \cdot \partial_x^i \frac{1}{\tau(x)} \Big|_{x=0}. \end{aligned}$$

It follows that

$$\text{span}\{w(0; z), \partial_x w(0; z), \dots\} = \text{span}\{\langle -1 | \alpha_1^k \psi(z) e^A | 0 \rangle\}_{k \geq 0}.$$

Note $\langle -1 | \alpha_1^k \psi(z) e^A | 0 \rangle$ is the inner product of $\alpha_{-1}^k \psi_{-1/2} | 0 \rangle$ with $\psi(z) e^A | 0 \rangle$. We have:

$$\begin{aligned}
\psi(z) e^A | 0 \rangle &= \sum_{r \in \mathbb{Z} + 1/2} z^{-r-1/2} \psi_r \cdot e^{\sum_{m,n \geq 0} A_{m,n} \psi_{-m-1/2} \psi_{-n-1/2}^*} | 0 \rangle \\
&= \sum_{k \geq 0} z^k \psi_{-k-1/2} e^A | 0 \rangle - \sum_{m,n \geq 0} z^{-n-1} A_{m,n} \psi_{-m-1/2} e^A | 0 \rangle \\
&= \sum_{m \geq 0} (z^m - \sum_{n \geq 0} A_{m,n} z^{-n-1}) \psi_{-m-1/2} e^{\sum_{m,n \geq 0} A_{m,n} \psi_{-m-1/2} \psi_{-n-1/2}^*} | 0 \rangle.
\end{aligned}$$

Now it is clear that:

$$(154) \quad \langle -1 | \psi(z) e^A | 0 \rangle = 1 - \sum_{n=0}^{\infty} A_{0,n} z^{-n-1}.$$

Next recall:

$$\alpha_{-1} = \frac{1}{2} \psi_{-1/2} \psi_{-1/2}^* + \sum_{n=1}^{\infty} (\psi_{-n-1/2} \psi_{n-1/2}^* - \psi_{-n-1/2}^* \psi_{n-1/2}),$$

and so one has

$$\alpha_{-1} \psi_{-1/2} | 0 \rangle = \psi_{-3/2} | 0 \rangle,$$

it follows that

$$\langle -1 | \alpha_1 \psi(z) e^A | 0 \rangle = z - \sum_{n=0}^{\infty} A_{1,n} z^{-n-1}.$$

Next from

$$\alpha_{-1}^2 \psi_{-1/2} | 0 \rangle = \psi_{-5/2} | 0 \rangle + \frac{1}{2} \psi_{-1/2} \psi_{-1/2}^* \psi_{-3/2} | 0 \rangle,$$

one gets:

$$\begin{aligned}
\langle -1 | \alpha_1^2 \psi(z) e^A | 0 \rangle &= z^2 - \sum_{n=0}^{\infty} A_{2,n} z^{-n-1} \\
&- \frac{1}{2} A_{1,0} (1 - \sum_{n=0}^{\infty} A_{0,n} z^{-1-n}) + \frac{1}{2} A_{0,0} (z - \sum_{n=0}^{\infty} A_{1,n} z^{-1-n}).
\end{aligned}$$

In general, from

$$\alpha_1^k \psi_{-1/2} | 0 \rangle = \psi_{-k-1/2} | 0 \rangle + \cdots,$$

one sees that

$$\langle -1 | \alpha_1^k \psi(z) e^A | 0 \rangle = (z^k - \sum_{n=0}^{\infty} A_{k,n} z^{-n-1}) + \cdots,$$

where \dots stand for terms with degrees lower than k . This completes the proof. \square

4.6. Fay identities. In last subsection we have seen that the wave function $w(\mathbf{T}; z)$ determines the tau-function $\tau(\mathbf{T})$, hence it also determines the free energy $F(\mathbf{T})$, and therefore, it should also determine the n -point function

$$(155) \quad \mathcal{F}(\xi_1, \dots, \xi_n; \mathbf{T}) = \nabla(\xi_1) \cdots \nabla(\xi_n) F(\mathbf{T}),$$

where

$$(156) \quad \nabla(\xi) = \sum_{n \geq 1} \xi^{-n-1} \frac{\partial}{\partial T_n}.$$

Let us show how this can be achieved. From the definition of the wave-function, we have

$$(157) \quad \frac{\tau(\mathbf{T} - [1/z])}{\tau(\mathbf{T})} = \exp\left(-\sum_{n=1} T_n z^n\right) \cdot w(\mathbf{T}; z).$$

Change \mathbf{T} to $\mathbf{T} + [1/\tilde{z}]$:

$$\begin{aligned} \frac{\tau(\mathbf{T} - [1/z] + [1/\tilde{z}])}{\tau(\mathbf{T} + [1/\tilde{z}])} &= \exp\left(-\sum_{n=1} \left(T_n + \frac{1}{n\tilde{z}^n}\right) z^n\right) \cdot w(\mathbf{T} + [1/\tilde{z}]; z) \\ &= \frac{\tilde{z}}{\tilde{z} - z} \exp\left(-\sum_{n=1} T_n z^n\right) \cdot w(\mathbf{T} + [1/\tilde{z}]; z). \end{aligned}$$

Take $\lim_{\tilde{z} \rightarrow z} \nabla_z$:

$$\begin{aligned} \frac{\nabla(z) \tau(\mathbf{T})}{\tau(\mathbf{T} + [1/z])} &= \lim_{\tilde{z} \rightarrow z} \partial_z \left(\frac{\tilde{z}}{\tilde{z} - z} \exp\left(-\sum_{n=1} T_n z^n\right) \cdot w(\mathbf{T} + [1/\tilde{z}]; z) \right) \\ &= \exp\left(-\sum_{n=1} T_n z^n\right) \cdot \lim_{\tilde{z} \rightarrow z} \frac{\tilde{z}}{(\tilde{z} - z)^2} \left(w(\mathbf{T} + [1/\tilde{z}]; z) \right. \\ &\quad - (\tilde{z} - z) \sum_{n=1} n T_n z^n \cdot w(\mathbf{T} + [1/\tilde{z}]; z) \\ &\quad \left. + (\tilde{z} - z) \cdot \partial_z w(\mathbf{T} + [1/\tilde{z}]; z) \right) \\ &= \exp\left(-\sum_{n=1} T_n z^n\right) \cdot z \lim_{\tilde{z} \rightarrow z} \left(\partial_{\tilde{z}}^2 w(\mathbf{T} + [1/\tilde{z}]; z) \right. \\ &\quad - 2 \sum_{n=1} n T_n z^n \cdot \partial_z w(\mathbf{T} + [1/\tilde{z}]; z) \\ &\quad \left. + 2 \partial_{\tilde{z}} \partial_z w(\mathbf{T} + [1/\tilde{z}]; z) \right). \end{aligned}$$

So we obtain a formula of the form:

$$\begin{aligned} \nabla(z)F(\mathbf{T}) &= w^*(\mathbf{T}; z) \cdot z \lim_{\tilde{z} \rightarrow z} \left(\partial_{\tilde{z}}^2 w(\mathbf{T} + [1/\tilde{z}]; z) \right. \\ &\quad \left. - 2 \sum_{n=1} n T_n z^n \cdot \partial_{\tilde{z}} w(\mathbf{T} + [1/\tilde{z}]; z) + 2 \partial_{\tilde{z}} \partial_z w(\mathbf{T} + [1/\tilde{z}]; z) \right). \end{aligned}$$

To get n -point functions, one can consider $\tau(\mathbf{T} - [1/\xi_1] + [1/\eta_1] + \cdots - [1/\xi_n] + [1/\eta_n])$. The result of this approach will involve both the wave-function and the dual wave function. This leads us to consider the product

$$(158) \quad w(\mathbf{T}; \xi) w^*(\mathbf{T}; \eta) = e^{\sum_{n \geq 1} T_n (\xi^n - \eta^n)} \cdot \frac{\tau(\mathbf{T} - [1/\xi]) \tau(\mathbf{T} + [1/\eta])}{\tau(\mathbf{T})^2}$$

and the Wronskian

$$\{w(\mathbf{T}; \xi), w^*(\mathbf{T}; \eta)\} = \begin{vmatrix} w(\mathbf{T}; \xi) & w^*(\mathbf{T}; \eta) \\ \partial_x w(\mathbf{T}; \xi) & \partial_x w^*(\mathbf{T}; \eta) \end{vmatrix},$$

and their restrictions to the diagonal $\xi = \eta$. Since

$$\begin{aligned} \partial_x w(\mathbf{T}; z) &= z \exp\left(\sum_{n=1} T_n z^n\right) \frac{\tau_U(\mathbf{T} - [1/z])}{\tau(\mathbf{T})} \\ &+ \exp\left(\sum_{n=1} T_n z^n\right) \frac{\partial_x \tau_U(\mathbf{T} - [1/z])}{\tau(\mathbf{T})} \\ &- \exp\left(\sum_{n=1} T_n z^n\right) \frac{\tau_U(\mathbf{T} - [1/z]) \cdot \partial_x \tau_U(\mathbf{T})}{\tau(\mathbf{T})^2} \\ &= \exp\left(\sum_{n=1} T_n z^n\right) \frac{1}{\tau(\mathbf{T})^2} \\ &\quad \cdot \left(z \tau_U(\mathbf{T}) \tau_U(\mathbf{T} - [1/z]) - \{\tau_U(\mathbf{T}), \tau_U(\mathbf{T}) - [1/z]\} \right), \end{aligned}$$

and dually,

$$\begin{aligned}
\partial_x w^*(\mathbf{T}; z) &= -z \exp\left(-\sum_{n=1} T_n z^n\right) \frac{\tau_U(\mathbf{T} + [1/z])}{\tau(\mathbf{T})} \\
&+ \exp\left(-\sum_{n=1} T_n z^n\right) \frac{\partial_x \tau_U(\mathbf{T} + [1/z])}{\tau(\mathbf{T})} \\
&- \exp\left(-\sum_{n=1} T_n z^n\right) \frac{\tau_U(\mathbf{T} + [1/z]) \cdot \partial_x \tau_U(\mathbf{T})}{\tau(\mathbf{T})^2} \\
&= -\exp\left(\sum_{n=1} T_n z^n\right) \frac{1}{\tau(\mathbf{T})^2} \\
&\quad \cdot \left(z \tau_U(\mathbf{T}) \tau_U(\mathbf{T} + [1/z]) - \{\tau_U(\mathbf{T}), \tau_U(\mathbf{T}) + [1/z]\} \right),
\end{aligned}$$

and so

$$\begin{aligned}
\{w(\mathbf{T}; \xi), w^*(\mathbf{T}; \eta)\} &= \exp \sum_{n \geq 1} T_n (\xi^n - \eta^n) \\
(159) \quad &\cdot \left(\frac{\{\tau(\mathbf{T} - [1/\xi]), \tau(\mathbf{T} + [1/\eta])\}}{\tau(\mathbf{T})^2} \right. \\
&\quad \left. - (\xi + \eta) \cdot \frac{\tau(\mathbf{T} - [1/\xi]) \tau(\mathbf{T} + [1/\eta])}{\tau(\mathbf{T})^2} \right).
\end{aligned}$$

So we need to consider the product and the Wronskian of $\tau(\mathbf{T} - [1/\xi])$ and $\tau(\mathbf{T} + [1/\eta])$. They can be studied using Fay identity for τ -function due to Sato, again one has to double the number of variables first then take suitable limits:

$$\begin{aligned}
&(s_0 - s_1)(s_2 - s_3) \tau(\mathbf{T} + [s_0] + [s_1]) \tau(\mathbf{T} + [s_2] + [s_3]) \\
(160) \quad &+ (s_0 - s_2)(s_3 - s_1) \tau(\mathbf{T} + [s_0] + [s_2]) \tau(\mathbf{T} + [s_3] + [s_1]) \\
&+ (s_0 - s_3)(s_1 - s_2) \tau(\mathbf{T} + [s_0] + [s_3]) \tau(\mathbf{T} + [s_1] + [s_2]) = 0.
\end{aligned}$$

For its derivation from the Hirota bilinear relations and its relation with Fay trisecant identity for theta functions, see e.g. [24]. In [1] this formula was used to derive the following formula:

$$(161) \quad w^*(\mathbf{T}; \eta) w(\mathbf{T}; \xi) = \frac{1}{\xi - \eta} \partial_x \left(\frac{X(\mathbf{T}; \xi, \eta) \tau(\mathbf{T})}{\tau(\mathbf{T})} \right).$$

This was done as follows. Take ∂_{s_0} on both sides of (160) and take $s_0 = s_3 = 0$, one can get:

$$\begin{aligned}
&\{\tau(\mathbf{T} + [s_1]), \tau(\mathbf{T} + [s_2])\} \\
(162) \quad &= \left(\frac{1}{s_2} - \frac{1}{s_1} \right) (\tau(\mathbf{T} + [s_1]) \tau(\mathbf{T} + [s_2]) - \tau(\mathbf{T}) \tau(\mathbf{T} + [s_1] + [s_2])).
\end{aligned}$$

This is the differential Fay identity. By changing \mathbf{T} to $\mathbf{T} - [s_2]$, one gets the following version [1, (3.11)]:

$$(163) \quad \begin{aligned} & \{\tau(\mathbf{T}), \tau(\mathbf{T} + [s_1] - [s_2])\} \\ &= (s_2^{-1} - s_1^{-1})(\tau(\mathbf{T} + [s_1] - [s_2])\tau(\mathbf{T}) - \tau(\mathbf{T} + [s_1])\tau(\mathbf{T} - [s_2])). \end{aligned}$$

Using this one gets by a computation similar to that of $\partial_x w(\mathbf{T}; z)$:

$$\begin{aligned} & \frac{1}{\xi - \eta} \partial_x \left(\frac{X(\mathbf{T}; \xi, \eta) \tau(\mathbf{T})}{\tau(\mathbf{T})} \right) \\ &= \frac{1}{\xi - \eta} \partial_x \left(\exp\left(\sum_{n \geq 1} T_n(\xi^n - \eta^n)\right) \cdot \frac{\tau(\mathbf{T} - [1/\xi] + [1/\eta])}{\tau(\mathbf{T})} \right) \\ &= \frac{\exp(\sum_{n \geq 1} T_n(\xi^n - \eta^n))}{(\xi - \eta) \tau(\mathbf{T})^2} \left((\xi - \eta) \cdot \tau(\mathbf{T} - [1/\xi] + [1/\eta]) \cdot \tau(\mathbf{T}) \right. \\ & \quad \left. - \{\tau(\mathbf{T}), \tau(\mathbf{T} - [1/\xi] + [1/\eta])\} \right) \\ &= \frac{\exp(\sum_{n \geq 1} T_n(\xi^n - \eta^n))}{\tau(\mathbf{T})^2} \tau(\mathbf{T} - [1/\xi]) \tau(\mathbf{T} + [1/\eta]) \\ &= w^*(\mathbf{T}; \eta) w(\mathbf{T}; \xi). \end{aligned}$$

This proves (161). By (159) and (162),

$$(164) \quad \begin{aligned} & \{w(\mathbf{T}; \xi), w^*(\mathbf{T}; \eta)\} \\ &= -(\xi + \eta) \cdot \exp \sum_{n \geq 1} T_n(\xi^n - \eta^n) \cdot \frac{\tau(\mathbf{T} - [1/\xi] + [1/\eta])}{\tau(\mathbf{T})}. \end{aligned}$$

In particular, after taking $\lim_{\eta \rightarrow \xi}$ on both sides of (164):

$$(165) \quad \{w(\mathbf{T}; \xi), w^*(\mathbf{T}; \xi)\} = -2\xi.$$

In the same fashion one can show that

$$(166) \quad \begin{aligned} & \{w(\mathbf{T}; \xi), w(\mathbf{T}; \eta)\} \\ &= -(\xi - \eta) \cdot \exp \sum_{n \geq 1} T_n(\xi^n + \eta^n) \cdot \frac{\tau(\mathbf{T} - [1/\xi] - [1/\eta])}{\tau(\mathbf{T})}, \end{aligned}$$

and for the dual wave-functions,

$$(167) \quad \begin{aligned} & \{w^*(\mathbf{T}; \xi), w^*(\mathbf{T}; \eta)\} \\ &= (\xi - \eta) \cdot \exp \sum_{n \geq 1} T_n(-\xi^n - \eta^n) \cdot \frac{\tau(\mathbf{T} + [1/\xi] + [1/\eta])}{\tau(\mathbf{T})}. \end{aligned}$$

4.7. A formula for bosonic one-point function. Taking $\lim_{\xi \rightarrow \eta}$ on both sides of (161):

$$(168) \quad w^*(\mathbf{T}; \xi)w(\mathbf{T}; \xi) = \partial_x \left(\frac{(\sum_{n \geq 1} nT_n \xi^{n-1} + \sum_{n \geq 1} \xi^{-n-1} \frac{\partial}{\partial T_n})\tau(\mathbf{T})}{\tau(\mathbf{T})} \right).$$

Write $\tau(\mathbf{T}) = \exp F(\mathbf{T})$. Then one has

$$(169) \quad w^*(\mathbf{T}; \xi)w(\mathbf{T}; \xi) = 1 + \partial_x \sum_{n \geq 1} w^{-n-1} \frac{\partial}{\partial T_n} F(\mathbf{T}),$$

or maybe it is more appropriate to write it as

$$(170) \quad w^*(\mathbf{T}; \xi)w(\mathbf{T}; \xi) = \partial_x \left(\sum_{n \geq 1} nT_n w^{n-1} + \sum_{n \geq 1} w^{-n-1} \frac{\partial}{\partial T_n} F(\mathbf{T}) \right).$$

One then formally has

$$(171) \quad \sum_{n \geq 1} nT_n w^{n-1} + \sum_{n \geq 1} w^{-n-1} \frac{\partial}{\partial T_n} F(\mathbf{T}) = \partial_x^{-1} (w^*(\mathbf{T}; \xi)w(\mathbf{T}; \xi)).$$

It is actually possible to express the left-hand side in terms of wave-function and the dual wave-function without taking the integral. Take ∂_ξ on both sides of (164):

$$\begin{aligned} & \{\partial_\xi w(\mathbf{T}; \xi), w^*(\mathbf{T}; \eta)\} \\ &= -\exp \sum_{n \geq 1} T_n (\xi^n - \eta^n) \cdot \frac{\tau(\mathbf{T} - [1/\xi] + [1/\eta])}{\tau(\mathbf{T})} \\ &- (\xi + \eta) \cdot \exp \sum_{n \geq 1} T_n (\xi^n - \eta^n) \\ &\cdot \frac{1}{\tau(\mathbf{T})} \left(\sum_{n \geq 1} nT_n \xi^{n-1} + \sum_{n \geq 1} \xi^{-n-1} \frac{\partial}{\partial T_n} \right) \tau(\mathbf{T} - [1/\xi] + [1/\eta]). \end{aligned}$$

In the above we have used the following identity:

$$(172) \quad \begin{aligned} & \partial_z w(\mathbf{T}; z) \\ &= \exp \left(\sum_{n=1} T_n z^n \right) \frac{\sum_{n=1} (nT_n z^n + z^{-n-1} \frac{\partial}{\partial T_n}) \tau(\mathbf{T} - [1/z])}{\tau(\mathbf{T})}. \end{aligned}$$

And so after taking $\lim_{\eta \rightarrow \xi}$, one gets:

$$(173) \quad \begin{aligned} & \{\partial_\xi w(\mathbf{T}; \xi), w^*(\mathbf{T}; \xi)\} \\ &= -1 - 2\xi \cdot \left(\sum_{n \geq 1} nT_n \xi^{n-1} + \sum_{n \geq 1} \xi^{-n-1} \frac{\partial}{\partial T_n} F(\mathbf{T}) \right). \end{aligned}$$

Similarly, take ∂_η on both sides of (164):

$$\begin{aligned}
& \{w(\mathbf{T}; \xi), \partial_\eta w^*(\mathbf{T}; \eta)\} \\
&= -\exp \sum_{n \geq 1} T_n (\xi^n - \eta^n) \cdot \frac{\tau(\mathbf{T} - [1/\xi] + [1/\eta])}{\tau(\mathbf{T})} \\
&+ (\xi + \eta) \cdot \exp \sum_{n \geq 1} T_n (\xi^n - \eta^n) \\
&\cdot \left(\sum_{n \geq 1} n T_n \eta^{n-1} + \sum_{n \geq 1} \eta^{-n-1} \frac{\partial}{\partial T_n} \right) \frac{\tau(\mathbf{T} - [1/\xi] + [1/\eta])}{\tau(\mathbf{T})},
\end{aligned}$$

and so after taking $\lim_{\eta \rightarrow \xi}$, one gets:

$$\begin{aligned}
& \{w(\mathbf{T}; \xi), \partial_\xi w^*(\mathbf{T}; \xi)\} \\
(174) \quad &= -1 + 2\xi \cdot \left(\sum_{n \geq 1} n T_n \xi^{n-1} + \sum_{n \geq 1} \xi^{-n-1} \frac{\partial}{\partial T_n} F(\mathbf{T}) \right).
\end{aligned}$$

Reformulating the above results, we get

Theorem 4.3. *For a τ -function $\tau(\mathbf{T})$ of the KP hierarchy, the following identities hold:*

$$\begin{aligned}
& \sum_{n \geq 1} n T_n \xi^{n-1} + \nabla(\xi) F(\mathbf{T}) \\
(175) \quad &= -\frac{1}{2\xi} (\{\partial_\xi w(\mathbf{T}; \xi), w^*(\mathbf{T}; \xi)\} + 1)
\end{aligned}$$

$$(176) \quad = \frac{1}{2\xi} (\{w(\mathbf{T}; \xi), \partial_\xi w^*(\mathbf{T}; \xi)\} + 1)$$

$$(177) \quad = \frac{1}{4\xi} (\{w(\mathbf{T}; \xi), \partial_\xi w^*(\mathbf{T}; \xi)\} - \{\partial_\xi w(\mathbf{T}; \xi), w^*(\mathbf{T}; \xi)\}).$$

In the case of KdV hierarchy, we recover [6, Theorem 1.2] by (177).

4.8. Higher Fay identities and bosonic n -point function of KP hierarchy in terms of fermionic one-point functions. To obtain general formula for bosonic n -point functions in terms of fermionic one-point functions, one can repeatedly apply the loop operators of the following form to the formula for the $(n-1)$ -point functions:

$$(178) \quad \nabla(\xi) = \sum_{n \geq 1} \xi^{-n-1} \frac{\partial}{\partial T_n}.$$

In the case of KdV hierarchy, one first sets $T_{2n} = 0$ and then applies instead the operator

$$\sum_{n \geq 1} \xi^{-2n-2} \frac{\partial}{\partial T_{2n-1}},$$

since the tau-function is independent of T_{2n} 's. Then following [6] one can derive their Theorem 1.7, which is a formula for bosonic n -point function. Let

$$(179) \quad \Theta(\xi; \mathbf{T}) = \frac{1}{2} \begin{pmatrix} -(ww^*)_x & -2ww^* \\ 2w_x w_x^* & (ww^*)_x \end{pmatrix},$$

then

$$(180) \quad \sum_{j_1 \geq 1} \frac{\partial F}{\partial T_{j_1}} \xi^{-2j_1-2} = \frac{1}{2} \text{Tr} \Theta(\xi; \mathbf{T}) - \sum_{j \geq 0} T_{2j+1} z^{2j+1},$$

and for $n \geq 2$,

$$(181) \quad \begin{aligned} & \sum_{j_1, \dots, j_n \geq 1} \frac{\partial F}{\partial T_{j_1} \dots \partial T_{j_n}} z_1^{-2j_1-2} \dots z_n^{-2j_n-2} \\ &= -\frac{1}{n} \sum_{\sigma \in S_n} \frac{\text{Tr}(\Theta(z_{\sigma(1)}) \dots \Theta(z_{\sigma(n)}))}{\prod_{i=1}^n (z_{\sigma(i)}^2 - z_{\sigma(i+1)}^2)} - \delta_{n,2} \frac{z_1^2 + z_2^2}{(z_1^2 - z_2^2)^2}. \end{aligned}$$

Write $R = R(\xi; \mathbf{T}) = w(\mathbf{T}; \xi) w^*(\mathbf{T}; \xi)$, then one can also write Θ as follows:

$$(182) \quad \Theta(\xi; \mathbf{T}) = \frac{1}{2} \begin{pmatrix} -R_x & -2R \\ R_{xx} - 2(\xi^2 - 2u)R & R_x \end{pmatrix}.$$

In fact R is a generating series of the Gelfand-Dickey polynomials. See [31] for a different proof of the formula of Bertola-Dubrovin-Yang [6] from this point of view.

The Bertola-Dubrovin-Yang formula suggests the possibility for general KP hierarchy to express the bosonic n -point function in terms of the wave-function and the dual wave-function and their derivatives in x and ξ_1, \dots, ξ_n . Since $\nabla(\xi_i)$ commute with ∂_x and ∂_{ξ_j} ($j \neq i$), it reduces

to computing $\nabla(\xi)w(\mathbf{T}; \eta)$ and $\nabla(\xi)w^*(\mathbf{T}; \eta)$. One has

$$\begin{aligned}
\nabla(\xi)w(\mathbf{T}; \eta) &= \sum_{m \geq 1} \xi^{-m-1} \frac{\partial}{\partial T_m} \left(\exp\left(\sum_{n=1} T_n \eta^n\right) \frac{\tau(\mathbf{T} - [1/\eta])}{\tau(\mathbf{T})} \right) \\
&= \sum_{m \geq 1} \xi^{-m-1} \eta^m \cdot \exp\left(\sum_{n=1} T_n \eta^n\right) \frac{\tau(\mathbf{T} - [1/\eta])}{\tau(\mathbf{T})} \\
&\quad + \exp\left(\sum_{n=1} T_n \eta^n\right) \frac{\nabla(\xi)\tau(\mathbf{T} - [1/\eta])}{\tau(\mathbf{T})} \\
&\quad - \exp\left(\sum_{n=1} T_n \eta^n\right) \frac{\tau(\mathbf{T} - [1/\eta])}{\tau(\mathbf{T})^2} \cdot \nabla(\xi)\tau(\mathbf{T}) \\
&= \left(\frac{1}{\xi - \eta} - \nabla(\xi)F(\mathbf{T}) \right) \cdot w(\mathbf{T}; \eta) \\
&\quad + \exp\left(\sum_{n=1} T_n \eta^n\right) \frac{\nabla(\xi)\tau(\mathbf{T} - [1/\eta])}{\tau(\mathbf{T})},
\end{aligned}$$

and similarly,

$$\begin{aligned}
\nabla(\xi)w^*(\mathbf{T}; \eta) &= \sum_{m \geq 1} \xi^{-m-1} \frac{\partial}{\partial T_m} \left(\exp\left(-\sum_{n=1} T_n \eta^n\right) \frac{\tau(\mathbf{T} + [1/\eta])}{\tau(\mathbf{T})} \right) \\
&= \left(-\frac{1}{\xi - \eta} - \nabla(\xi)F(\mathbf{T}) \right) \cdot w^*(\mathbf{T}; \eta) \\
&\quad + \exp\left(-\sum_{n=1} T_n \eta^n\right) \frac{\nabla(\xi)\tau(\mathbf{T} + [1/\eta])}{\tau(\mathbf{T})}.
\end{aligned}$$

Now note

$$\frac{\nabla(\xi)\tau(\mathbf{T} \pm [1/\eta])}{\tau(\mathbf{T})} = \lim_{\tilde{\xi} \rightarrow \xi} \partial_{\tilde{\xi}} \frac{\tau(\mathbf{T} - [1/\xi] + [1/\tilde{\xi}] \pm [1/\eta])}{\tau(\mathbf{T})}.$$

In [2], the following generalization of (166) has been proved:

$$\begin{aligned}
&\{w(\mathbf{T}; \xi_1), \dots, w(\mathbf{T}; \xi_k)\} \\
(183) \quad &= \prod_{1 \leq i < j \leq k} (\xi_j - \xi_i) \cdot \exp \sum_{n \geq 1} T_n \sum_{i=1}^k \xi_i^n \cdot \frac{\tau(\mathbf{T} - \sum_{i=1}^k [1/\xi_i])}{\tau(\mathbf{T})},
\end{aligned}$$

We conjecture the following two identities should hold:

$$\begin{aligned}
&\{w^*(\mathbf{T}; \eta_1), \dots, w^*(\mathbf{T}; \eta_l)\} \\
(184) \quad &= \prod_{1 \leq i < j \leq l} (\eta_i - \eta_j) \cdot \exp \sum_{n \geq 1} (-T_n \sum_{i=1}^l \eta_i^n) \cdot \frac{\tau(\mathbf{T} + \sum_{i=1}^l [1/\eta_i])}{\tau(\mathbf{T})},
\end{aligned}$$

$$\begin{aligned}
& \{w(\mathbf{T}; \xi_1), \dots, w(\mathbf{T}; \xi_k), w^*(\mathbf{T}; \eta_1), \dots, w^*(\mathbf{T}; \eta_l)\} \\
&= \prod_{1 \leq i < j \leq k} (\xi_j - \xi_i) \cdot \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq l}} (-\xi_i - \eta_j) \cdot \prod_{1 \leq i < j \leq l} (\eta_i - \eta_j) \\
(185) \quad & \cdot \exp \sum_{n \geq 1} T_n \left(\sum_{i=1}^k \xi_i^n - \sum_{i=1}^l \eta_i^n \right) \\
& \cdot \frac{\tau(\mathbf{T} - \sum_{i=1}^k [1/\xi_i] + \sum_{i=1}^l [1/\eta_i])}{\tau(\mathbf{T})}.
\end{aligned}$$

In particular, we conjecture the following identities to hold:

$$\begin{aligned}
& \frac{\tau(\mathbf{T} - [1/\xi_1] - [1/\xi_2] + [1/\eta_1])}{\tau(\mathbf{T})} \\
&= \exp \sum_{n \geq 1} T_n (-\xi_1^n - \xi_2^n + \eta_1^n) \cdot \frac{1}{\xi_2 - \xi_1} \cdot \frac{1}{(-\xi_1 - \eta_1)(-\xi_2 - \eta_1)} \\
& \cdot \{w(\mathbf{T}; \xi_1), w(\mathbf{T}; \xi_2), w^*(\mathbf{T}; \eta_1)\},
\end{aligned}$$

$$\begin{aligned}
& \frac{\tau(\mathbf{T} - [1/\xi_1] + [1/\eta_1] + [1/\eta_2])}{\tau(\mathbf{T})} \\
&= \exp \sum_{n \geq 1} T_n (-\xi_1^n + \eta_1^n + \eta_2^n) \cdot \frac{1}{(-\xi_1 - \eta_1)(-\xi_1 - \eta_2)} \cdot \frac{1}{\eta_1 - \eta_2} \\
& \cdot \{w(\mathbf{T}; \xi_1), w^*(\mathbf{T}; \eta_1), w^*(\mathbf{T}; \eta_2)\}.
\end{aligned}$$

Assuming these identities, one can get

$$\begin{aligned}
& \frac{\nabla(\xi) \tau(\mathbf{T} + [1/\eta])}{\tau(\mathbf{T})} = \lim_{\tilde{\xi} \rightarrow \xi} \partial_{\tilde{\xi}} \frac{\tau(\mathbf{T} - [1/\tilde{\xi}] + [1/\tilde{\xi}] + [1/\eta])}{\tau(\mathbf{T})} \\
&= \exp \left(\sum_{n \geq 1} T_n \eta^n \right) \left(\frac{1}{2\xi(\xi^2 - \eta^2)} \left(- \sum_{n \geq 1} n T_n \xi^{n-1} + \partial_{\xi} \right) \right. \\
& \left. - \frac{3\xi + \eta}{4\xi^2(\xi + \eta)^2(\xi - \eta)} \right) \{w(\mathbf{T}; \xi), w^*(\mathbf{T}; \xi), w^*(\mathbf{T}; \eta)\},
\end{aligned}$$

$$\begin{aligned}
\frac{\nabla(\xi)\tau(\mathbf{T} - [1/\eta])}{\tau(\mathbf{T})} &= \lim_{\tilde{\xi} \rightarrow \xi} \partial_{\tilde{\xi}} \frac{\tau(\mathbf{T} - [1/\xi] + [1/\tilde{\xi}] - [1/\eta])}{\tau(\mathbf{T})} \\
&= -\exp\left(-\sum_{n \geq 1} T_n \eta^n\right) \left(\frac{1}{2\xi(\xi^2 - \eta^2)} \left(-\sum_{n \geq 1} n T_n \xi^{n-1} + \partial_{\xi}\right) \right. \\
&\quad \left. - \frac{3\xi - \eta}{4\xi^2(\xi - \eta)^2(\xi + \eta)} \right) \{w(\mathbf{T}; \xi), w(\mathbf{T}; \eta), w^*(\mathbf{T}; \xi)\},
\end{aligned}$$

With these identities, one can express $\nabla(\xi)w(\mathbf{T}; \eta)$ and $\nabla(\xi)w^*(\mathbf{T}; \eta)$ in terms of the Wronskians of the wave-function and the dual wave-functions and their derivatives in ξ .

5. BOSONIC N-POINT FUNCTIONS IN TERMS OF NORMALIZED BASES AND ADMISSIBLE BASES

In last Section we have discussed the possibility of expressing the bosonic n -point functions in terms of the fermionic 1-point functions for general KP hierarchy. In this Section we will derive some formulas for bosonic n -point functions in terms of a fermionic two-point function, and the latter in term of an admissible basis or a normalized basis.

5.1. Bosonic n -point functions and connected bosonic n -point functions. From the point of view of bosonic conformal field theory, one can consider the bosonic n -point functions:

$$(186) \quad \langle \alpha(\xi_1) \cdots \alpha(\xi_n) \rangle_U := \langle 0 | e^{\sum_{n \geq 1} T_n \alpha_n} \alpha(\xi_1) \cdots \alpha(\xi_n) | U \rangle / \tau_U.$$

Since we have

$$\begin{aligned}
\alpha(\xi_i)\alpha(\xi_j) &= \alpha(\xi_i)_+\alpha(\xi_j)_+ + \alpha(\xi_i)_-\alpha(\xi_j)_+ \\
&\quad + \alpha(\xi_j)_-\alpha(\xi_i)_+ + \alpha(\xi_i)_-\alpha(\xi_j)_- + i_{\xi_i, \xi_j} \frac{1}{(\xi_i - \xi_j)^2} \\
&= \alpha(\xi_j)\alpha(\xi_i),
\end{aligned}$$

the n -point function $\langle \alpha(\xi_1) \cdots \alpha(\xi_n) \rangle_U$ is symmetric with respect to positions ξ_1, \dots, ξ_n , so we will write it as $f(\xi_1, \dots, \xi_n)$. In the above we have used the following notation:

$$(187) \quad i_{x,y} \frac{1}{(x-y)^n} = \sum_{k \geq 0} \binom{-n}{k} x^{-n-k} y^k.$$

One can also define the connected n -point functions $\langle \alpha(\xi_1) \cdots \alpha(\xi_n) \rangle_U^c$, and write it as $f^c(\xi_1, \dots, \xi_n)$. These two kinds of n -point functions are

related to each other by Möbius inversion:

$$(188) \quad f(\xi_1, \dots, \xi_n) = \sum_{I_1 \amalg \dots \amalg I_k = [n]} f^c(\xi_{I_1}) \cdots f^c(\xi_{I_k}),$$

$$(189) \quad f^c(\xi_1, \dots, \xi_n) = \sum_{I_1 \amalg \dots \amalg I_k = [n]} (-1)^{k-1} (k-1)! f(\xi_{I_1}) \cdots f(\xi_{I_k}),$$

where $[n]$ is the index set $\{1, \dots, n\}$, by $I_1 \amalg \dots \amalg I_k = [n]$ we mean a partition of $[n]$ into nonempty subsets I_1, \dots, I_k , and $\xi_{I_i} = (\xi_j)_{j \in I_i}$.

Write $\tau_U = e^{F_U}$. Note

$$(190) \quad \alpha(\xi) = \sum_{n \geq 1} \xi^{-n-1} \frac{\partial}{\partial T_n} + \sum_{n \geq 1} n T_n \xi^{n-1}.$$

Then the bosonic one-point function is given by

$$(191) \quad f^c(\xi_1) = \langle \alpha(\xi) \rangle_U = \sum_{n \geq 1} \frac{\partial F_U}{\partial T_n} \xi^{-n-1} + \sum_{n \geq 1} n T_n \xi^{n-1}.$$

Similarly, the bosonic two-point function is given by:

$$\begin{aligned} \langle \alpha(\xi_1) \alpha(\xi_2) \rangle_U &= \sum_{n_1, n_2 \geq 1} \left(\frac{\partial^2 F_U}{\partial T_{n_1} \partial T_{n_2}} + \frac{\partial F_U}{\partial T_{n_1}} \frac{\partial F_U}{\partial T_{n_2}} \right) \xi_1^{-n_1-1} \xi_2^{-n_2-1} \\ &+ \sum_{n=1}^{\infty} n \xi_1^{-n-1} \xi_2^{n-1} \\ &+ \sum_{n_1 \geq 1} n_1 T_{n_1} \xi_1^{n_1-1} \cdot \sum_{n_2 \geq 1} \frac{\partial F_U}{\partial T_{n_2}} \xi_2^{n_2-1} \\ &+ \sum_{n_1 \geq 1} \frac{\partial F_U}{\partial T_{n_1}} \xi_1^{n_1-1} \cdot \sum_{n_2 \geq 1} n_2 T_{n_2} \xi_2^{n_2-1} \\ &+ \sum_{n_1 \geq 1} n_1 T_{n_1} \xi_1^{n_1-1} \cdot \sum_{n_2 \geq 1} n_2 T_{n_2} \xi_2^{n_2-1}. \end{aligned}$$

It can be rewritten as

$$(192) \quad \begin{aligned} \langle \alpha(\xi_1) \alpha(\xi_2) \rangle_U &= \frac{1}{(\xi_1 - \xi_2)^2} + \sum_{n_1, n_2 \geq 1} \frac{\partial^2 F_U}{\partial T_{n_1} \partial T_{n_2}} \xi_1^{-n_1-1} \xi_2^{-n_2-1} \\ &+ \langle \alpha(\xi_1) \rangle_U \cdot \langle \alpha(\xi_2) \rangle_U. \end{aligned}$$

From this we get

$$(193) \quad f^c(\xi_1, \xi_2) = \frac{1}{(\xi_1 - \xi_2)^2} + \sum_{n_1, n_2 \geq 1} \frac{\partial^2 F_U}{\partial T_{n_1} \partial T_{n_2}} \xi_1^{-n_1-1} \xi_2^{-n_2-1}.$$

Proposition 5.1. *For $m \geq 3$,*

$$(194) \quad f^c(\xi_1, \dots, \xi_m) = \sum_{n_1, \dots, n_m \geq 1} \frac{\partial^m F_U}{\partial T_{n_1} \dots \partial T_{n_m}} \xi_1^{-n_1-1} \dots \xi_m^{-n_m-1}.$$

Proof. This can be proved by induction. Denote by $\tilde{f}^c(\xi_1, \dots, \xi_m)$ the right-hand side of (191) or (193) or (194), then one can see that

$$(195) \quad e^{-F_U} \alpha(\xi_{m+1}) e^{F_U} = \tilde{f}^c(\xi_1),$$

and

$$(196) \quad \alpha(\xi_{m+1})_+ \tilde{f}^c(\xi_1, \dots, \xi_m) = \tilde{f}^c(\xi_1, \dots, \xi_{m+1}).$$

Suppose that one has $f^c(\xi_1, \dots, \xi_k) = \tilde{f}^c(\xi_1, \dots, \xi_k)$ for $k \leq m$, then one has:

$$\begin{aligned} f(\xi_1, \dots, \xi_{m+1}) &= e^{-F_U} \alpha(\xi_{m+1}) (f(\xi_2, \dots, \xi_m) e^{F_U}) \\ &= e^{-F_U} \alpha(\xi_{m+1}) (e^{F_U} \sum_{I_1 \amalg \dots \amalg I_k = [n]} \tilde{f}^c(\xi_{I_1}) \dots \tilde{f}^c(\xi_{I_k})) \\ &= e^{-F_U} \alpha(\xi_{m+1}) e^{F_U} \cdot \sum_{I_1 \amalg \dots \amalg I_k = [n]} \tilde{f}^c(\xi_{I_1}) \dots \tilde{f}^c(\xi_{I_k}) \\ &\quad + \alpha(\xi_{m+1})_+ \sum_{I_1 \amalg \dots \amalg I_k = [n]} \tilde{f}^c(\xi_{I_1}) \dots \tilde{f}^c(\xi_{I_k}) \\ &= \tilde{f}^c(\xi_{m+1}) \cdot \sum_{I_1 \amalg \dots \amalg I_k = [n]} \tilde{f}^c(\xi_{I_1}) \dots \tilde{f}^c(\xi_{I_k}) \\ &\quad + \sum_{I_1 \amalg \dots \amalg I_k = [n]} \sum_{j=1}^k \tilde{f}^c(\xi_{I_1}) \dots \tilde{f}^c(\xi_{m+1}, \xi_{I_j}) \dots \tilde{f}^c(\xi_{I_k}). \end{aligned}$$

By induction hypothesis and (188),

$$f^c(\xi_1, \dots, \xi_{m+1}) = \tilde{f}^c(\xi_1, \dots, \xi_{m+1}).$$

This finishes the proof. \square

5.2. Higher Fay identities and bosonic n -point function of KP hierarchy in terms of fermionic two-point functions. Let us recall Okounkov's approach [20] to higher Fay identities developed by Adler-Shioda-van Moerbeke [2]. Consider

$$(197) \quad \begin{aligned} &\langle \psi(\xi_1) \dots \psi(\xi_m) \psi^*(\eta_1) \dots \psi^*(\eta_n) \rangle_U \\ &= \langle 0 | \psi(\xi_1) \dots \psi(\xi_m) \psi^*(\eta_1) \dots \psi^*(\eta_n) | U \rangle. \end{aligned}$$

This can be computed in several different ways. First one can combine

$$(198) \quad \psi(\xi) \psi^*(\eta) = \frac{1}{\xi - \eta} \Gamma_-(\{\xi\} - \{\eta\}) \Gamma_+(\{\eta^{-1}\} - \{\xi^{-1}\}),$$

with

$$(199) \quad \Gamma_+(\{\mathbf{T}\})\Gamma_-(\{\mathbf{S}\}) = e^{\sum_{n \geq 1} n T_n S_n} \Gamma_-(\{\mathbf{S}\})\Gamma_+(\{\mathbf{T}\})$$

to get

$$(200) \quad \begin{aligned} & \langle \psi(\xi_1) \cdots \psi(\xi_m) \psi^*(\eta_n) \cdots \psi^*(\eta_1) \rangle_U \\ &= \frac{\Delta_n(\xi) \Delta_n(\eta)}{\prod_{1 \leq i, j \leq n} (\xi_i - \eta_j)} \tau_U \left(\sum_{i=1}^n (\{\eta_i^{-1}\} - \{\xi_i^{-1}\}) \right). \end{aligned}$$

In particular, when $n = 1$,

$$(201) \quad \langle \psi(\xi) \psi^*(\eta) \rangle_U = \frac{1}{\xi - \eta} \tau_U(\{\eta^{-1}\} - \{\xi^{-1}\}).$$

On the other hand, by Wick's Theorem,

$$(202) \quad \langle \psi(\xi_1) \cdots \psi(\xi_n) \psi^*(\eta_n) \cdots \psi^*(\eta_1) \rangle_U = \det(\langle \psi(\xi_i) \psi^*(\eta_j) \rangle_U)_{1 \leq i, j \leq n}.$$

Combining the preceding three identities:

$$(203) \quad \begin{aligned} & \det \left(\frac{1}{\xi_i - \eta_j} \tau_U(\{\eta_j^{-1}\} - \{\xi_i^{-1}\}) \right) \\ &= \frac{\Delta_n(\xi) \Delta_n(\eta)}{\prod_{1 \leq i, j \leq n} (\xi_i - \eta_j)} \tau_U \left(\sum_{i=1}^n (\{\eta_i^{-1}\} - \{\xi_i^{-1}\}) \right). \end{aligned}$$

This is a special case of the following higher Fay identities developed by Adler-Shioda-van Moerbeke [2, (45)]:

$$(204) \quad \begin{aligned} & \det \left(\frac{1}{\xi_i - \eta_j} \frac{\tau_U(\mathbf{T} + \{\eta_j^{-1}\} - \{\xi_i^{-1}\})}{\tau_U(\mathbf{T})} \right) \\ &= \frac{\Delta_n(\xi) \Delta_n(\eta)}{\prod_{1 \leq i, j \leq n} (\xi_i - \eta_j)} \frac{\tau_U(\sum_{i=1}^n (\{\eta_i^{-1}\} - \{\xi_i^{-1}\}))}{\tau_U(\mathbf{T})}. \end{aligned}$$

5.3. Fermionic two-point function in terms of affine coordinates. Recall $U = e^A|0\rangle$, where

$$(205) \quad A = \sum_{m, n \geq 0} a_{m, n} \psi_{-m-1/2} \psi_{-n-1/2}^*.$$

It is easy to see that

$$(206) \quad \langle \psi(\xi) \psi^*(\eta) \rangle_U = i_{\xi, \eta} \frac{1}{\xi - \eta} + \sum_{m, n \geq 0} a_{m, n} \xi^{-m-1} \eta^{-n-1}.$$

In particular, $\langle \psi(\xi) \psi^*(\eta) \rangle_U$ contains the same information as the operator A . Write

$$(207) \quad A(\xi, \eta) = \sum_{m, n \geq 0} a_{m, n} \xi^{-m-1} \eta^{-n-1}.$$

Then we have

$$(208) \quad \langle \psi(\xi) \psi^*(\eta) \rangle_U = i_{\xi, \eta} \frac{1}{\xi - \eta} + A(\xi, \eta).$$

5.4. Bosonic n -point function in terms of fermionic two-point functions. Use the OPE:

$$(209) \quad \psi(\xi) \psi^*(\eta) = i_{\xi, \eta} \frac{1}{\xi - \eta} + : \psi(\xi) \psi^*(\eta) :$$

one gets:

$$(210) \quad \langle : \psi(\xi) \psi^*(\eta) : \rangle_U = A(\xi, \eta).$$

After taking $\lim_{\eta \rightarrow \xi}$:

$$(211) \quad \langle \alpha(\xi) \rangle_U = A(\xi, \xi).$$

Let us now compute $\langle \alpha(\xi_1) \alpha(\xi_2) \rangle_U$ in the same fashion. On the one hand we have:

$$\begin{aligned} & \langle \psi(\xi_1) \psi^*(\eta_1) \psi(\xi_2) \psi^*(\eta_2) \rangle_U \\ &= -\langle \psi(\xi_1) \psi(\xi_2) \psi^*(\eta_1) \psi^*(\eta_2) \rangle_U \\ &+ \left(i_{\xi_2, \eta_1} \frac{1}{\xi_2 - \eta_1} - i_{\eta_1, \xi_2} \frac{1}{\xi_2 - \eta_1} \right) \cdot \langle \psi(\xi_1) \psi^*(\eta_2) \rangle_U \\ &= \begin{vmatrix} \langle \psi(\xi_1) \psi^*(\eta_1) \rangle_U & \langle \psi(\xi_1) \psi^*(\eta_2) \rangle_U \\ \langle \psi(\xi_2) \psi^*(\eta_1) \rangle_U & \langle \psi(\xi_2) \psi^*(\eta_2) \rangle_U \end{vmatrix} \\ &+ \left(i_{\xi_2, \eta_1} \frac{1}{\xi_2 - \eta_1} - i_{\eta_1, \xi_2} \frac{1}{\xi_2 - \eta_1} \right) \cdot (i_{\xi_1, \eta_2} \frac{1}{\xi_1 - \eta_2} + A(\xi_1, \eta_2)) \\ &= \begin{vmatrix} i_{\xi_1, \eta_1} \frac{1}{\xi_1 - \eta_1} + A(\xi_1, \eta_1) & i_{\xi_1, \eta_2} \frac{1}{\xi_1 - \eta_2} + A(\xi_1, \eta_2) \\ i_{\xi_2, \eta_1} \frac{1}{\xi_2 - \eta_1} + A(\xi_2, \eta_1) & i_{\xi_2, \eta_2} \frac{1}{\xi_2 - \eta_2} + A(\xi_2, \eta_2) \end{vmatrix} \\ &+ \left(i_{\xi_2, \eta_1} \frac{1}{\xi_2 - \eta_1} - i_{\eta_1, \xi_2} \frac{1}{\xi_2 - \eta_1} \right) \cdot (i_{\xi_1, \eta_2} \frac{1}{\xi_1 - \eta_2} + A(\xi_1, \eta_2)) \\ &= \begin{vmatrix} i_{\xi_1, \eta_1} \frac{1}{\xi_1 - \eta_1} + A(\xi_1, \eta_1) & i_{\xi_1, \eta_2} \frac{1}{\xi_1 - \eta_2} + A(\xi_1, \eta_2) \\ i_{\eta_1, \xi_2} \frac{1}{\xi_2 - \eta_1} + A(\xi_2, \eta_1) & i_{\xi_2, \eta_2} \frac{1}{\xi_2 - \eta_2} + A(\xi_2, \eta_2) \end{vmatrix}. \end{aligned}$$

In the last step we use the following identity for determinants of 2×2 -matrices:

$$(212) \quad \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} + \begin{vmatrix} 0 & c_{12} \\ b_{21} & c_{22} \end{vmatrix} = \begin{vmatrix} c_{11} & c_{12} \\ b_{12} + c_{21} & c_{22} \end{vmatrix}.$$

On the other hand,

$$\begin{aligned}
& \langle \psi(\xi_1) \psi^*(\eta_1) \psi(\xi_2) \psi^*(\eta_2) \rangle_U \\
&= \left\langle \left(i_{\xi_1, \eta_1} \frac{1}{\xi_1 - \eta_1} + : \psi(\xi_1) \psi^*(\eta_1) : \right) \left(i_{\xi_2, \eta_2} \frac{1}{\xi_2 - \eta_2} + : \psi(\xi_2) \psi^*(\eta_2) : \right) \right\rangle_U \\
&= i_{\xi_1, \eta_1} \frac{1}{\xi_1 - \eta_1} \cdot i_{\xi_2, \eta_2} \frac{1}{\xi_2 - \eta_2} + i_{\xi_1, \eta_1} \frac{1}{\xi_1 - \eta_1} \cdot A(\xi_2, \eta_2) \\
&+ i_{\xi_2, \eta_2} \frac{1}{\xi_2 - \eta_2} \cdot A(\xi_1, \eta_1) + \langle : \psi(\xi_1) \psi^*(\eta_1) :: \psi(\xi_2) \psi^*(\eta_2) : \rangle_U.
\end{aligned}$$

Combining the two calculations, one gets:

$$\begin{aligned}
& \langle : \psi(\xi_1) \psi^*(\eta_1) :: \psi(\xi_2) \psi^*(\eta_2) : \rangle_U \\
&= \begin{vmatrix} A(\xi_1, \eta_1) & i_{\xi_1, \eta_2} \frac{1}{\xi_1 - \eta_2} + A(\xi_1, \eta_2) \\ i_{\eta_1, \xi_2} \frac{1}{\xi_2 - \eta_1} + A(\xi_2, \eta_1) & A(\xi_2, \eta_2) \end{vmatrix}.
\end{aligned}$$

Now we take $\lim_{\eta_1 \rightarrow \xi_1} \lim_{\eta_2 \rightarrow \xi_2}$:

$$\langle \alpha(\xi_1) \alpha(\xi_2) \rangle_U = \begin{vmatrix} A(\xi_1, \xi_1) & i_{\xi_1, \xi_2} \frac{1}{\xi_1 - \xi_2} + A(\xi_1, \xi_2) \\ i_{\xi_1, \xi_2} \frac{1}{\xi_2 - \xi_1} + A(\xi_2, \xi_1) & A(\xi_2, \xi_2) \end{vmatrix}.$$

Now we generalize the computation for $\langle \psi(\xi_1) \psi^*(\eta_1) \psi(\xi_2) \psi^*(\eta_2) \rangle_U$ to the case of $\langle \psi(\xi_1) \psi^*(\eta_1) \cdots \psi(\xi_n) \psi^*(\eta_n) \rangle_U$ for $n > 2$. One first moves all $\psi^*(\eta_j)$'s to the right of all $\psi(\xi_i)$'s, and then apply (202). By an analogue of (212), one gets:

$$(213) \quad \langle \psi(\xi_1) \psi^*(\eta_1) \cdots \psi(\xi_n) \psi^*(\eta_n) \rangle_U = \det(C_{ij})_{1 \leq i, j \leq n},$$

where

$$(214) \quad C_{ij} = \begin{cases} i_{\xi_i, \eta_j} \frac{1}{\xi_i - \eta_j} + A(\xi_i, \eta_j), & i \leq j, \\ i_{\eta_j, \xi_i} \frac{1}{\xi_i - \eta_j} + A(\xi_i, \eta_j), & i > j. \end{cases}$$

On the other hand,

$$\begin{aligned}
& \langle \psi(\xi_1) \psi^*(\eta_1) \cdots \psi(\xi_n) \psi^*(\eta_n) \rangle_U \\
&= \left\langle \left(i_{\xi_1, \eta_1} \frac{1}{\xi_1 - \eta_1} + : \psi(\xi_1) \psi^*(\eta_1) : \right) \cdots \left(i_{\xi_n, \eta_n} \frac{1}{\xi_n - \eta_n} + : \psi(\xi_n) \psi^*(\eta_n) : \right) \right\rangle_U.
\end{aligned}$$

So we get an identity of the form

$$\begin{aligned}
& \left\langle \left(i_{\xi_1, \eta_1} \frac{1}{\xi_1 - \eta_1} + : \psi(\xi_1) \psi^*(\eta_1) : \right) \cdots \left(i_{\xi_n, \eta_n} \frac{1}{\xi_n - \eta_n} + : \psi(\xi_n) \psi^*(\eta_n) : \right) \right\rangle_U \\
&= \det(C_{ij})_{1 \leq i, j \leq n}.
\end{aligned}$$

It is clear that the left-hand side can be expanded into 2^n terms, among which only one term does not contain a factor of the form $i_{\xi_i, \eta_i} \frac{1}{\xi_i - \eta_i}$, that is $\langle : \psi(\xi_1) \psi^*(\eta_1) : \cdots : \psi(\xi_n) \psi^*(\eta_n) : \rangle_U$. Similarly, the right-hand can also be expanded into 2^n term, this can be done by decomposing

each row vector of the matrix into a sum of two row vectors, one of them is given by $(\delta_{i,j} i_{\xi_i, \eta_i} \frac{1}{\xi_i - \eta_i})_{j=1, \dots, n}$. And so each of the 2^n terms is a determinant, and there is only one term that does not contain a factor of the form $i_{\xi_i, \eta_i} \frac{1}{\xi_i - \eta_i}$, that is given by $\det(\hat{C}_{i,j}(\xi_i, \eta_j))_{1 \leq i, j \leq n}$, where

$$(215) \quad \hat{C}_{ij} = \begin{cases} A(\xi_i, \xi_i), & i = j, \\ C_{i,j}, & i \neq j. \end{cases}$$

Therefore,

$$(216) \quad \langle : \psi(\xi_1) \psi^*(\eta_1) : \cdots : \psi(\xi_n) \psi^*(\eta_n) : \rangle_U = \det(\hat{C}_{i,j})_{1 \leq i, j \leq n}.$$

After taking $\lim_{\eta_i \rightarrow \xi_i}$, we obtain:

$$(217) \quad \langle \alpha(\xi_1) \cdots \alpha(\xi_n) \rangle_U = \det(\hat{A}(\xi_i, \xi_j))_{1 \leq i, j \leq n},$$

where

$$(218) \quad \hat{A}(\xi_i, \xi_j) = \begin{cases} i_{\xi_i, \xi_j} \frac{1}{\xi_i - \xi_j} + A(\xi_i, \xi_j), & i < j, \\ A(\xi_i, \xi_i), & i = j, \\ i_{\xi_j, \xi_i} \frac{1}{\xi_i - \xi_j} + A(\xi_i, \xi_j), & i > j. \end{cases}$$

5.5. Connected bosonic n -point function in terms of fermionic two-point functions. We now prove a combinatorial result.

Proposition 5.2. *Suppose that we have two sequences of functions $\{\varphi(\xi_1, \dots, \xi_n)\}_{n \geq 1}$ and $\{\varphi^c(\xi_1, \dots, \xi_n)\}_{n \geq 1}$, related to each other by Möbius inversion:*

$$(219) \quad \varphi(\xi_1, \dots, \xi_n) = \sum_{I_1 \amalg \cdots \amalg I_k = [n]} \varphi^c(\xi_{I_1}) \cdots \varphi^c(\xi_{I_k}),$$

$$(220) \quad \varphi^c(\xi_1, \dots, \xi_n) = \sum_{I_1 \amalg \cdots \amalg I_k = [n]} (-1)^{k-1} (k-1)! \varphi(\xi_{I_1}) \cdots \varphi(\xi_{I_k}),$$

where the summations are taken over partitions of $[n]$ into nonempty subsets I_1, \dots, I_k , and $\xi_{I_i} = (\xi_j)_{j \in I_i}$. Suppose that there are functions $B(\xi, \eta)$ such that

$$(221) \quad \varphi(\xi_1, \dots, \xi_n) = \det(B(\xi_i, \xi_j))_{1 \leq i, j \leq n}.$$

Then one has

$$(222) \quad \varphi^c(\xi_1, \dots, \xi_n) = (-1)^{n-1} \sum_{n\text{-cycles } \sigma} \prod_{i=1}^n B(\xi_{\sigma(i)} \xi_{\sigma(i+1)}),$$

where the summations are taken over n -cycles σ , and $\sigma(n+1) = \sigma(1)$.

Proof. This can be proved by induction. When $n = 1$, it holds automatically. Suppose that it holds for $1, \dots, n-1$. By (221),

$$(223) \quad \varphi(\xi_1, \dots, \xi_n) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n B(\xi_i, \xi_{\sigma(i)}).$$

Now we rewrite the right-hand side by writing a permutation $\sigma \in S_n$ as a product of the disjoint cycles. If σ is an n -cycle, then $\text{sign}(\sigma) = (-1)^{n-1}$, and so the contribution of all n -cycles is exactly the right-hand side of (222), which we denote by $\psi(\xi_1, \dots, \xi_n)$. Therefore, one gets:

$$(224) \quad \varphi(\xi_1, \dots, \xi_n) = \sum_{I_1 \amalg \dots \amalg I_k = [n]} \psi(\xi_{I_1}) \cdots \psi(\xi_{I_k}).$$

The proof is completed by using (219) and the induction hypothesis. \square

As a corollary,

Theorem 5.3. *For $n \geq 2$,*

$$(225) \quad \begin{aligned} & \sum_{j_1, \dots, j_n \geq 1} \left. \frac{\partial^n F_U}{\partial T_{j_1} \cdots \partial T_{j_n}} \right|_{\mathbf{T}=0} \xi_1^{-j_1-1} \cdots \xi_n^{-j_n-1} \\ &= (-1)^{n-1} \sum_{n\text{-cycles}} \prod_{i=1}^n \hat{A}(\xi_{\sigma(i)}, \xi_{\sigma(i+1)}). \end{aligned}$$

5.6. Fermionic two-point function in terms of admissible basis.

By combining (201) with (209) and (210), one gets:

$$(226) \quad A(\xi, \eta) = \frac{1}{\xi - \eta} \tau_U(\{\eta^{-1}\} - \{\xi^{-1}\}) - i_{\xi, \eta} \frac{1}{\xi - \eta}.$$

Let us now compute $\tau_U(\{\eta^{-1}\} - \{\xi^{-1}\})$ in terms of admissible basis. We first note that

$$(227) \quad \tau_U(\{\eta^{-1}\} - \{\xi^{-1}\}) = 1 + (\xi - \eta)A(\xi, \eta),$$

and so $\tau_U(\{\eta^{-1}\} - \{\xi^{-1}\})$ is of order at most one in $A_{m,n}$'s. Recall that

$$\tau_U(\{\eta^{-1}\} - \{\xi^{-1}\}) = \langle 0 | \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{\eta^n} - \frac{1}{\xi^n} \right) \alpha_n \right) e^A | 0 \rangle$$

is equal to the inner product of $e^A|0\rangle$ with $\exp\left(\sum_{n=1}^{\infty} \frac{1}{n}(\frac{1}{\eta^n} - \frac{1}{\xi^n})\alpha_{-n}\right)|0\rangle$.

On the other hand one has an expansion:

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n}(\frac{1}{\eta^n} - \frac{1}{\xi^n})\alpha_{-n}\right)|0\rangle = \sum_{\mu} s_{\mu}(p_k = \frac{1}{\eta^k} - \frac{1}{\xi^k}, k \geq 1) \cdot |\mu\rangle,$$

and

$$e^A|0\rangle = |0\rangle + \sum_{m,n \geq 0} A_{m,n} \psi_{-m-1/2} \psi_{-n-1/2}^* |0\rangle + \cdots,$$

where \cdots are higher order terms in $A_{m,n}$, therefore by (227), one gets

$$(228) \quad s_{\mu}(p_k = \frac{1}{\eta^k} - \frac{1}{\xi^k}, k \geq 1) = 0$$

except for $\mu = \emptyset$ or $(m|n)$ for some $m, n \geq 0$, and

$$\begin{aligned} & \sum_{m,n \geq 0} s_{(m|n)}(p_k = \frac{1}{\eta^k} - \frac{1}{\xi^k}, k \geq 1) \cdot (-1)^n A_{n,m} \\ &= (\xi - \eta) \sum_{m,n \geq 0} A_{m,n} \xi^{-m-1} \eta^{-n-1}, \end{aligned}$$

and so

$$(229) \quad s_{(m|n)}(p_k = \frac{1}{\eta^k} - \frac{1}{\xi^k}, k \geq 1) = (-1)^n (\xi - \eta) \xi^{-n-1} \eta^{-m-1}.$$

Therefore,

$$\begin{aligned} (230) \quad & \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}(\frac{1}{\eta^n} - \frac{1}{\xi^n})\alpha_{-n}\right)|0\rangle \\ &= |0\rangle + \sum_{m,n \geq 0} (\xi - \eta) \xi^{-n-1} \eta^{-m-1} \psi_{-m-1/2} \psi_{-n-1/2}^* |0\rangle. \end{aligned}$$

Suppose now that U is specified by an admissible basis $\{f_n = z^{n+1/2} + \sum_{k < n} c_{n,k} z^{k+1/2}\}_{n \geq 0}$, then by (69),

$$\begin{aligned}
 & \tau_U(\{\eta^{-1}\} - \{\xi^{-1}\}) \\
 &= 1 + \sum_{m,n \geq 0} (\xi - \eta) \xi^{-n-1} \eta^{-m-1} \\
 (231) \quad & \cdot \begin{vmatrix} c_{0,-m-1} & 1 & & & \\ c_{1,-m-1} & c_{1,0} & 1 & & \\ c_{2,-m-1} & c_{2,0} & c_{2,1} & 1 & \\ \vdots & \vdots & \vdots & \vdots & \\ c_{n-1,m-1} & c_{n-1,0} & c_{n-1,1} & \cdots & 1 \\ c_{n,m-1} & c_{n,0} & c_{n,1} & \cdots & c_{n,n-1} \end{vmatrix}.
 \end{aligned}$$

As a corollary,

$$(232) \quad a_{n,m} = \begin{vmatrix} c_{0,-m-1} & 1 & & & \\ c_{1,-m-1} & c_{1,0} & 1 & & \\ c_{2,-m-1} & c_{2,0} & c_{2,1} & 1 & \\ \vdots & \vdots & \vdots & \vdots & \\ c_{n-1,m-1} & c_{n-1,0} & c_{n-1,1} & \cdots & 1 \\ c_{n,m-1} & c_{n,0} & c_{n,1} & \cdots & c_{n,n-1} \end{vmatrix}.$$

Of course this last identity can be obtained more directly by comparing the coefficients of $\psi_{-m-1/2} \psi_{-n-1/2}^*$ on both sides of

$$(233) \quad |U\rangle = e^A |0\rangle.$$

The approach that we take in this Section relate $A(\xi, \eta)$ to $\tau_U(\{\eta^{-1}\} - \{\xi^{-1}\})$ through some specialization of the Schur functions. In Section 6.9 we will see an important application of this idea.

6. EMERGENCE OF THE AIRY FUNCTION IN TOPOLOGICAL 2D GRAVITY

In this Section we specialize the results in the preceding Sections to the case of topological 2D gravity.

6.1. Reductions. Let A be a subalgebra of $\mathbb{C}((z^{-1}))$ such that

$$(234) \quad A \cap \mathbb{C}[[z^{-1}]] = \mathbb{C},$$

and define the A -reduced Sato Grassmannian by:

$$(235) \quad \text{Gr}_{(0)}^A := \{U \in \text{Gr}_{(0)} \mid f(s_1)U \subset U, \forall f \in A\}.$$

Then for $U \in \text{Gr}_{(0)}^A$, τ_U is a tau-function of the KP hierarchy, such that the associated pseudo-differential operator L satisfies

$$(236) \quad f(L) \text{ is a differential operator, for all } f \in A.$$

The Airy curve $y = \frac{1}{2}x^2$ defines a reduction to the KdV hierarchy by taking $A = \mathbb{C}[z^2]$.

6.2. Noncommutative deformation of the Airy curve. A consequence of Sato's approach to KP hierarchy is the emergence of the noncommutative deformation theory of the Airy curve. Quantize the Airy curve by understanding p as $i\frac{\partial}{\partial x}$, and rewrite the equation of the Airy curve as a differential operator $P_0 = \frac{\partial^2}{\partial x^2} + 2x$. By a noncommutative deformation of P_0 we mean a differential operator of the form:

$$(237) \quad P_{\mathbf{t}} = \frac{\partial^2}{\partial x^2} + 2u(x; \mathbf{t}).$$

Witten Conjecture/Kontsevich Theorem states that when $u(x; \mathbf{t}) = \frac{\partial^2 F(\mathbf{t})}{\partial t_0^2}$, $x = t_0$, the operator P_t satisfies the following evolution equations:

$$(238) \quad \partial_{t_n} P_t = (2n+1)!! \cdot [(P_{\mathbf{t}}^{(2n+1)/2})_+, P_{\mathbf{t}}].$$

6.3. Emergence of the Airy equation. The wave-function for the operator $P_0 = \partial_x^2 + 2x$ is a function $w(x; \xi)$ such that

$$(239) \quad \partial_x^2 w + 2x \cdot w = \xi^2 \cdot w.$$

Perform a change of variables:

$$(240) \quad \tilde{x} = 2^{-2/3}(\xi^2 - 2x).$$

Then one gets:

$$(241) \quad \partial_{\tilde{x}}^2 w = \tilde{x} \cdot w.$$

Recall the Airy functions $\text{Ai}(x)$ and $\text{Bi}(x)$ can be defined by the following integral representations:

$$(242) \quad \text{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt,$$

$$(243) \quad \text{Bi}(x) = \frac{1}{\pi} \int_0^\infty \left(\exp\left(-\frac{t^3}{3} + xt\right) + \sin\left(\frac{t^3}{3} + xt\right) \right) dt.$$

They are two solutions to the Airy equation

$$(244) \quad \partial_x^2 y - x \cdot y = 0.$$

These functions and their first derivatives admit the following asymptotic expansions:

$$(245) \quad \text{Ai}(x) \sim \frac{1}{2} \pi^{-1/2} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} \sum_{m=0}^{\infty} \frac{(6m-1)!!}{(6\sqrt{-2})^{2m}(2m)!} x^{-3m/2},$$

$$(246) \quad \text{Ai}'(x) \sim \frac{1}{2} \pi^{-1/2} x^{1/4} e^{-\frac{2}{3}x^{3/2}} \sum_{m=0}^{\infty} \frac{(6m-1)!!}{(6\sqrt{-2})^{2m}(2m)!} \frac{6m+1}{6m-1} x^{-3m/2},$$

$$(247) \quad \text{Bi}(x) \sim \pi^{-1/2} x^{-1/4} e^{\frac{2}{3}x^{3/2}} \sum_{m=0}^{\infty} \frac{(6m-1)!!}{(6\sqrt{2})^{2m}(2m)!} x^{-3m/2},$$

$$(248) \quad \text{Bi}'(x) \sim \pi^{-1/2} x^{1/4} e^{\frac{2}{3}x^{3/2}} \sum_{m=0}^{\infty} \frac{(6m-1)!!}{(6\sqrt{2})^{2m}(2m)!} \frac{6m+1}{6m-1} x^{-3m/2}.$$

6.4. Wave-function from the Airy function. There exists a suitably chosen $c(\xi) = 2^{5/6} \pi^{1/2} \xi^{1/2} e^{\xi^3/3}$ such that

$$\begin{aligned} w(x; \xi) &= c(\xi) \cdot \text{Ai}(\tilde{x}) \\ &= c(\xi) \cdot \frac{1}{2} \pi^{-1/2} \tilde{x}^{-1/4} e^{-\frac{2}{3}\tilde{x}^{3/2}} \sum_{m=0}^{\infty} \frac{(6m-1)!!}{(6\sqrt{-2})^{2m}(2m)!} \tilde{x}^{-3m/2} \\ &= c(\xi) \cdot \frac{1}{2} \pi^{-1/2} \cdot 2^{1/6} \cdot (\xi^2 - 2x)^{-1/4} e^{-\frac{1}{3}(\xi^2 - 2x)^{3/2}} \\ &\quad \cdot \sum_{m=0}^{\infty} (-1)^m \frac{(6m-1)!!}{6^{2m}(2m)!} (\xi^2 - 2x)^{-3m/2} \\ &= e^{x\xi} \cdot (1 - 2\xi^{-2}x)^{-1/4} e^{-\frac{1}{3}\xi^3(1-2\xi^{-2}x)^{3/2} - x\xi + \frac{1}{3}\xi^3} \\ &\quad \cdot \sum_{m=0}^{\infty} (-1)^m \frac{(6m-1)!!}{6^{2m}(2m)!} \xi^{-3m} (1 - 2\xi^{-2}x)^{-3m/2} \end{aligned}$$

is of the form

$$(249) \quad w(x; \xi) = e^{x\xi} \left(1 + \frac{b_1(x)}{\xi} + \frac{b_2(x)}{\xi^2} + \dots \right).$$

This is form of the wave-function at $t_k = 0$ for $k \geq 1$. Now we have:

$$(250) \quad w(0; \xi) = \sum_{m=0}^{\infty} (-1)^m \frac{(6m-1)!!}{6^{2m}(2m)!} \xi^{-3m},$$

$$(251) \quad \partial_x w(0; \xi) = \sum_{m=0}^{\infty} (-1)^{m+1} \frac{(6m-1)!!}{6^{2m}(2m)!} \frac{6m+1}{6m-1} \xi^{-3m+1}.$$

Denote these series by $c(\xi)$ and $q(\xi)$ respectively. They are called the Faber-Zagier series [21]. They are closely related to the asymptotic

series of $\text{Ai}(z^2)$ and $\text{Ai}'(z^2)$ respectively, and then one has

$$(252) \quad Dc(z) = q(z),$$

$$(253) \quad D^2c(z) = z^2 \cdot q(z),$$

where D is the differential operator:

$$(254) \quad D := z + \frac{1}{2z^2} - \frac{1}{z} \frac{\partial}{\partial z}.$$

6.5. Airy functions in Kac-Schwarz's geometric characterization of Z_{WK} . Consider the vector space W spanned by $\{z^{1/2}D^n c(z)\}_{n \geq 0}$. It is clear W lies in the $\text{Gr}_{(0)}$ and so W determines a tau-function τ_W of the KP hierarchy. Because it is clear that

$$(255) \quad z^2 W \subset W,$$

this tau-function τ_W is 2-reduced, i.e., it is a tau-function of the KdV hierarchy. It is also clear that

$$(256) \quad z^{1/2} D z^{-1/2} W \subset W.$$

Kac-Schwarz [12] showed that this is equivalent to τ_W satisfying the puncture equation:

$$(257) \quad L_{-1} \tau_W = 0.$$

Hence by Witten Conjecture/Kontsevich Theorem, τ_W is the partition function of the topological 2D gravity. Furthermore, by combining (255) and (256), one gets

$$(258) \quad z^{2n+1/2} D z^{-1/2} W \subset W, \quad n \geq 0.$$

In *loc. cite.* it was shown these imply that τ_W satisfies the Virasoro constraints:

$$(259) \quad L_{n-1} \tau_W = 0, \quad n \geq 0.$$

As pointed out by Looijenga [17], if one sets $\mathcal{L}_n = -\frac{1}{2} z^{2n+5/2} D z^{-1/2}$, $n \geq -1$, then

$$(260) \quad [\mathcal{L}_m, \mathcal{L}_n] = (m - n) \mathcal{L}_{m+n}.$$

6.6. Airy function in Kontsevich's proof of Witten Conjecture. Kontsevich [14] identified the generating series Z_{WK} of intersection numbers of ψ -classes on $\overline{\mathcal{M}}_{g,n}$ with τ_W , hence proved the Witten

Conjecture. (See [17] for an exposition.) He first established the Main Identity: For $g \geq 0$, $n \geq 1$ such that $2g - 2 + n > 0$,

$$(261) \quad \sum_{\sum_{i=1}^n m_i = 3g-3+n} \langle \tau_{m_1} \cdots \tau_{m_n} \rangle_g \prod_{i=1}^n \frac{(2m_i - 1)!!}{\lambda_i^{2m_i+1}} \\ = \sum_{\Gamma \in G_{g,n}} \frac{2^{-|V(\Gamma)|}}{|\text{Aut}(\Gamma)|} \prod_{e \in E(\Gamma)} \prod_{e \in E(\Gamma)} \frac{2}{\tilde{\lambda}(e)},$$

where $G_{g,n}$ is the set of isomorphism classes of connected trivalent ribbon graphs of genus g with n boundary cycles labeled by $1, \dots, n$. Given an edge e , it is a band with two boundaries, labeled by i and j , respectively, then

$$(262) \quad \tilde{\lambda}(e) = \lambda_i + \lambda_j.$$

Next, using the Main Identity and the Wick's Theorem, Kontsevich obtain the following matrix model for the free energy. Let

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$$

be a positive definite hermitian $N \times N$ -matrix, and let

$$(263) \quad t_i = t_i(\Lambda) = -(2i - 1)!! \cdot \text{Tr } \Lambda^{-(2i+1)}.$$

Then his Theorem 1.1 states the formal series $F(t_0(\Lambda), t_1(\Lambda), \dots)$ is an asymptotic expansion of $\log Z^{(n)}(\Lambda)$ when $\Lambda^{-1} \rightarrow 0$, where

$$(264) \quad Z^{(N)}(\Lambda) := c_\Lambda \int \exp \text{Tr} \left(\frac{\sqrt{-1}}{6} X^3 - \frac{1}{2} X^2 \Lambda \right),$$

where the constant c_Λ is chosen to be

$$c_\Lambda = \det \left(\frac{1}{4\pi} (\Lambda \otimes 1 + 1 \otimes \Lambda) \right)^{1/2}$$

so that

$$c_\Lambda \int \exp \text{Tr} \left(-\frac{1}{2} X^2 \Lambda \right) dX = 1.$$

Next, Harish-Chandra formula was used to the reduce the matrix integral of Airy form in Kontsevich model to a determinantal expression involves the Airy function and its derivatives. The relationship with Sato tau-function was established by [14, Lemma 4.1] to get

$$(265) \quad Z^{(N)}(\Lambda) = \frac{\det_{1 \leq i, j \leq N} (D^{j-1} a(\lambda_i^{-1}))}{\det_{1 \leq i, j \leq N} (\lambda_i^{1-j})}.$$

In the final step, Kontsevich showed that when $N \rightarrow \infty$, $\tau^{(N)}(\Lambda)$ gives τ_W . This gives an explicit construction of the Witten-Kontsevich tau-function in terms of the Airy function.

Let us note Airy functions Ai and Bi also appear in the setting of quantum spectral curves [27].

6.7. Explicit fermionic expressions of Witten-Kontsevich tau-function. The above formula for the Witten-Kontsevich tau-function was explained by Itzykson-Zuber [11]. Inspired by [12] and [14], the author [29] found an explicit expression of Z_{WK} as follows. After suitable linear change of coordinates in t_n 's:

$$(266) \quad t_n = (2n+1)!! T_{2n+1} = \frac{(2n+1)!!}{2n+1} p_{2n+1},$$

and the boson-fermion correspondence, the Witten-Kontsevich tau-function is a Bogoliubov transform in the fermionic picture:

$$(267) \quad Z_{WK} = e^A |0\rangle, \quad A = \sum_{m,n \geq 0} A_{m,n} \psi_{-m-\frac{1}{2}} \psi_{-n-\frac{1}{2}}^*,$$

where the coefficients $A_{m,n} = 0$ if $m+n \not\equiv -1 \pmod{3}$ and

$$\begin{aligned} A_{3m-1,3n} &= A_{3m-3,3n+2} = \frac{(-1)^n (6m+1)!!}{36^{m+n} (2(m+n))!} \\ &\quad \cdot \prod_{j=0}^{n-1} (m+j) \cdot \prod_{j=1}^n (2m+2j-1) \cdot (B_n(m) + \frac{b_n}{6m+1}), \\ A_{3m-2,3n+1} &= \frac{(-1)^{n+1} (6m+1)!!}{36^{m+n} (2(m+n))!} \\ &\quad \cdot \prod_{j=0}^{n-1} (m+j) \cdot \prod_{j=1}^n (2m+2j-1) \cdot (B_n(m) + \frac{b_n}{6m-1}), \end{aligned}$$

where $B_n(m)$ is a polynomial in m of degree $n-1$ defined by:

$$(268) \quad B_n(x) = \frac{1}{6} \sum_{j=1}^n 108^j b_{n-j} \cdot (x+n)_{[j-1]},$$

where

$$(269) \quad (a)_{[j]} = \begin{cases} 1, & j=0, \\ a(a-1) \cdots (a-j+1), & j>0, \end{cases}$$

and b_n is a constant depending on n defined by:

$$(270) \quad b_n = \frac{2^n \cdot (6n+1)!!}{(2n)!}.$$

The following are some examples of the coefficients $A_{m,n}$:

$$\begin{aligned} A_{3m-1,0} &= \frac{1}{36^m} \frac{(6m+1)!!}{(2m)!} \cdot \frac{1}{6m+1}, \\ A_{3m-2,1} &= -\frac{1}{36^m} \frac{(6m+1)!!}{(2m)!} \cdot \frac{1}{6m-1}, \\ A_{3m-3,2} &= \frac{1}{36^m} \frac{(6m+1)!!}{(2m)!} \cdot \frac{1}{6m+1}. \end{aligned}$$

6.8. Transition matrix from admissible basis to normalized basis. The above result was established by Virasoro constraints. Balogh and Yang [4] rederived it from the point of view of Sato's construction of the tau-function from admissible bases and normalized bases, so it becomes a special case of the general results for KP hierarchy developed in this paper. Let us briefly recall their beautiful results. Suppose that $W \in \text{Gr}_{(0)}$ is given by an admissible basis of the form $\{z^{2n+1/2}a(z), z^{2n+3/2}b(z)\}_{n \geq 0}$, where

$$a(z) = 1 + \sum_{n \geq 1} a_n z^{-n}, \quad b(z) = 1 + \sum_{n \geq 1} b_n z^{-n}.$$

Define a matrix $G(z) = \begin{pmatrix} G(z)_{11} & G(z)_{12} \\ G(z)_{21} & G(z)_{22} \end{pmatrix}$ whose entries are given by

$$\begin{aligned} G(z)_{11} &= \sum_{n \geq 0} a_{2n} z^{-n}, & G(z)_{12} &= \sum_{n \geq 0} b_{2n+1} z^{-n}, \\ G(z)_{21} &= \sum_{n \geq 1} a_{2n-1} z^{-n}, & G(z)_{22} &= \sum_{n \geq 0} b_{2n} z^{-n}. \end{aligned}$$

Suppose that the corresponding normalized basis is given by

$$(271) \quad z^n + \sum_{m \geq 0} A_{m,n} z^{-m-1}.$$

For $m, n \geq 0$, define a matrix $Z_{m,n}$ by

$$(272) \quad Z_{m,n} := \begin{pmatrix} A_{2m+1,2n} & A_{2m+1,2n+1} \\ A_{2m,2n} & A_{2m,2n+1} \end{pmatrix}.$$

Then [4, Theorem 1.1]:

$$(273) \quad \sum_{m,n=0}^{\infty} Z_{m,n} x^{-m-1} y^{-n-1} = \frac{1}{x-y} (I - G(x)G(y)^{-1}).$$

When $a(z)$ and $b(z)$ are given by the following series related to (250) and (251) respectively:

$$(274) \quad a(z) = \sum_{m=0}^{\infty} \frac{(6m-1)!!}{6^{2m}(2m)!} \xi^{-3m},$$

$$(275) \quad b(z) = - \sum_{m=0}^{\infty} \frac{(6m-1)!!}{6^{2m}(2m)!} \frac{6m+1}{6m-1} z^{-3m+1},$$

$A_{m,n}$ are given by the explicit expressions in last subsection. (This is when we use (266). If we use instead as in [4]

$$(276) \quad t_n = -(2n+1)!! T_{2n+1} = \frac{(2n+1)!!}{2n+1} p_{2n+1},$$

then $a(z)$ and $b(z)$ should be replaced by $c(z)$ and $q(z)$ respectively.) Furthermore, the matrix G is related to the R -matrix of in the theory of Frobenius manifold associated with 3-spin structures [26]. According to [21], when $a(z)$ and $b(z)$ are taken to be the Faber-Zagier series,

$$(277) \quad R(z) = \begin{pmatrix} \sum_{n \geq 0} b_{2n} z^{2n} & - \sum_{n \geq 0} b_{2n+1} z^{2n+1} \\ - \sum_{n \geq 0} a_{2n+1} z^{2n+1} & \sum_{n \geq 0} a_{2n} z^{2n} \end{pmatrix}.$$

Then [4, Theorem 1.2]:

$$(278) \quad R(z) = z^{\frac{1}{6}\sigma_3} G(z^{-2/3}) z^{-\frac{1}{6}\sigma_3},$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. See [5] for generalizations of these results to all r -spin cases.

6.9. Relationship with a formula of Okounkov. Let us now relate the formula (273) of Balogh-Yang [4] to a formula of Okounkov [20] used in a proof of the Witten Conjecture. Since $\det G(z) = 1$, one has

$$(279) \quad G(z) = \begin{pmatrix} \sum_{j \geq 0} a_{2j} z^{-3j} & \sum_{j \geq 0} b_{2j+1} z^{-3j-1} \\ \sum_{j \geq 0} a_{2j+1} z^{-3j-2} & \sum_{j \geq 0} b_{2j} z^{-3j} \end{pmatrix},$$

$$(280) \quad G(z)^{-1} = \begin{pmatrix} \sum_{j \geq 0} b_{2j} z^{-3j} & - \sum_{j \geq 0} b_{2j+1} z^{-3j-1} \\ - \sum_{j \geq 0} a_{2j+1} z^{-3j-2} & \sum_{j \geq 0} a_{2j} z^{-3j} \end{pmatrix}$$

From

$$(281) \quad \sum_{m,n=0}^{\infty} x^{-m-1} y^{-n-1} \begin{pmatrix} A_{2m+1,2n} & A_{2m+1,2n+1} \\ A_{2m,2n} & A_{2m,2n+1} \end{pmatrix} \\ = \frac{1}{x-y} (I - G(x)G(y)^{-1}),$$

one gets four identities:

$$\begin{aligned} \sum_{m,n \geq 0} A_{2m+1,2n} x^{-m-1} y^{-n-1} &= \frac{1}{x-y} (1 - \sum_{j \geq 0} a_{2j} x^{-3j} \cdot \sum_{k \geq 0} b_{2k} y^{-3k} \\ &\quad + \sum_{j \geq 0} b_{2j+1} x^{-3j-1} \cdot \sum_{k \geq 0} a_{2k+1} y^{-3k-2}), \\ \sum_{m,n \geq 0} A_{2m+1,2n+1} x^{-m-1} y^{-n-1} &= \frac{1}{x-y} (\sum_{j \geq 0} a_{2j} x^{-3j} \cdot \sum_{k \geq 0} b_{2k+1} y^{-3k-1} \\ &\quad - \sum_{j \geq 0} b_{2j+1} x^{-3j-1} \cdot \sum_{k \geq 0} a_{2k} y^{-3k}), \\ \sum_{m,n \geq 0} A_{2m,2n} x^{-m-1} y^{-n-1} &= \frac{1}{x-y} (-\sum_{j \geq 0} a_{2j+1} x^{-3j-2} \cdot \sum_{k \geq 0} b_{2k} y^{-3k} \\ &\quad + \sum_{j \geq 0} b_{2j} x^{-3j} \cdot \sum_{k \geq 0} a_{2k+1} y^{-3k-2}), \\ \sum_{m,n \geq 0} A_{2m,2n+1} x^{-m-1} y^{-n-1} &= \frac{1}{x-y} (1 + \sum_{j \geq 0} a_{2j+1} x^{-3j-2} \cdot \sum_{k \geq 0} b_{2k+1} y^{-3k-1} \\ &\quad - \sum_{j \geq 0} b_{2j} x^{-3j} \cdot \sum_{k \geq 0} a_{2k} y^{-3k}). \end{aligned}$$

They can be combined into one beautiful identity:

$$(282) \quad \sum_{m,n \geq 0} A_{m,n} x^{-m-1} y^{-n-1} \\ = \frac{1}{x-y} + \frac{1}{x^2-y^2} (a(x) \cdot b(-y) - a(-y) \cdot b(x)).$$

To relate this to [20], we now use (227) to get:

$$(283) \quad \sum_{m,n \geq 0} a_{m,n} x^{-m-1} y^{-n-1} = \frac{1}{x-y} + \frac{1}{x-y} \tau_W(\{y^{-1}\} - \{x^{-1}\}).$$

Because $\tau_W(\mathbf{T})$ depends only on T_{2n+1} , so we have

$$\begin{aligned}\tau_W(\{y^{-1}\} - \{x^{-1}\}) &= \tau_W(\{y^{-1}\} + \{-x^{-1}\}) \\ &= \langle 0 | \exp \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{y^n} + \frac{1}{(-x)^n} \right) \alpha_n | W \rangle,\end{aligned}$$

so it is the inner product of $|W\rangle$ with

$$\exp \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{y^n} + \frac{1}{(-x)^n} \right) \alpha_{-n} | 0 \rangle = \sum_{\mu} s_{\mu} (p_n = \frac{1}{y^n} + \frac{1}{(-x)^n}, n \geq 1) \cdot |\mu\rangle.$$

The specialization of the Schur functions can be computed by the Jacobi-Trudy identities: For $n \geq l(\mu)$,

$$(284) \quad s_{\mu} = \det(h_{\mu_i - i + j})_{1 \leq i, j \leq n},$$

and for $n \geq l(\mu^t)$,

$$(285) \quad s_{\mu} = \det(e_{\mu_i^t - i + j})_{1 \leq i, j \leq n}.$$

For the specialization with $p_n = y^{-n} + (-x)^{-n}$, it is clear that the generating series for elementary symmetric functions e_k and the complete symmetric functions h_k are given by

$$(286) \quad E(t) = (1 - x^{-1}t)(1 + y^{-1}t) = 1 + (y^{-1} - x^{-1})t - x^{-1}y^{-1}t^2,$$

and

$$(287) \quad H(t) = \frac{1}{(1 + x^{-1}t)(1 - y^{-1}t)} = \sum_{n \geq 0} \left(\sum_{i+j=n} x^{-i}y^{-j} \right) t^n$$

respectively. It follows that

$$e_1 = y^{-1} - x^{-1}, \quad e_2 = -x^{-1}y^{-1}, \quad e_k = 0, \quad k > 2,$$

and

$$(288) \quad h_n = \sum_{i+j=n} (-1)^i x^{-i} y^{-j} = \frac{(-x)^{-n-1} - y^{-n-1}}{(-x)^{-1} - y^{-1}}.$$

It follows from (285) that when $l(\mu) > 2$,

$$(289) \quad s_{\mu}(p_n = \frac{1}{y^n} + \frac{1}{(-x)^n}, n \geq 1) = 0.$$

Indeed, since $\mu_1^t = l(\mu) > 2$, one has $e_{\mu_1^t - 1 + j} = 0$ for $j = 1, \dots, n$, i.e., the first row in the determinant on the right-hand side of (285) all

vanish. Therefore, one only has to consider the case of $l(\mu) = 1$ and $l(\mu) = 2$, for which we use (284) to get

$$\begin{aligned}
s_{(m|0)}(p_k = \frac{1}{y^k} + \frac{1}{(-x)^k}, k \geq 1) &= h_{m+1} = \frac{(-x)^{-m-2} - y^{-m-2}}{(-x)^{-1} - y^{-1}}, \\
s_{(m|1)}(p_k = \frac{1}{y^k} + \frac{1}{(-x)^k}, k \geq 1) &= s_{(m+1,1)} = \frac{(-x)^{-m-2}y^{-1} - (-x)^{-1}y^{-m-2}}{(-x)^{-1} - y^{-1}}, \\
s_{(m_1, m_2)|(1,0)}(p_k = \frac{1}{y^k} + \frac{1}{(-x)^k}, k \geq 1) \\
&= s_{(m_1+1, m_2+2)}(p_k = \frac{1}{y^k} + \frac{1}{(-x)^k}, k \geq 1) \\
&= \begin{vmatrix} h_{m_1+1} & h_{m_1+2} \\ h_{m_2+1} & h_{m_2+2} \end{vmatrix} = \frac{(-x)^{-m_1-2}y^{-m_2-2} - (-x)^{-m_2-2}y^{-m_1-2}}{(-x)^{-1} - y^{-1}}.
\end{aligned}$$

So we get

$$\begin{aligned}
&\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{y^n} + \frac{1}{(-x)^n}\right) \alpha_{-n}\right) |0\rangle \\
&= |0\rangle + \sum_{m \geq 0} \frac{(-x)^{-m-2} - y^{-m-2}}{(-x)^{-1} - y^{-1}} \cdot \psi_{-m-1/2} \psi_{-1/2}^* |0\rangle \\
&- \sum_{m \geq 0} \frac{(-x)^{-m-2}y^{-1} - (-x)^{-1}y^{-m-2}}{(-x)^{-1} - y^{-1}} \cdot \psi_{-m-1/2} \psi_{-3/2}^* |0\rangle \\
&- \sum_{m_1 > m_2 \geq 0} \frac{(-x)^{-m_1-2}y^{-m_2-2} - (-x)^{-m_2-2}y^{-m_1-2}}{(-x)^{-1} - y^{-1}} \\
&\cdot \psi_{-m_1-1/2} \psi_{-3/2}^* \psi_{-m_2-1/2} \psi_{-1/2}^* |0\rangle.
\end{aligned}$$

Note we have

$$\begin{aligned}
&(z^{1/2} + \sum_{j \geq 1} a_j z^{-3j+1/2}) \wedge (z^{3/2} + \sum_{k \geq 1} b_k z^{-3k+3/2}) \\
&= z^{1/2} \wedge z^{3/2} - \sum_{k \geq 1} b_k z^{-3k+3/2} \wedge z^{1/2} + \sum_{j \geq 1} a_j z^{-3j+1/2} \wedge z^{3/2} \\
&+ \sum_{j \geq k \geq 1} a_j b_k z^{-3j+1/2} \wedge z^{-3k+3/2} - \sum_{1 \leq j < k} a_j b_k z^{-3k+3/2} \wedge z^{-3j+1/2},
\end{aligned}$$

so the relevant terms in $|W\rangle$ are

$$\begin{aligned}
|W\rangle &= |0\rangle + \sum_{j \geq 1} a_j \cdot \psi_{-3j+1/2} \psi_{-1/2}^* |0\rangle + \sum_{k \geq 1} b_k \cdot \psi_{-3k+3/2} \psi_{-3/2}^* |0\rangle \\
&+ \sum_{j \geq k \geq 1} a_j b_k \cdot \psi_{-3j+1/2} \psi_{-3k+3/2} \psi_{-3/2}^* \psi_{-1/2}^* |0\rangle \\
&- \sum_{1 \leq j < k} a_j b_k \cdot \psi_{-3k+3/2} \psi_{-3j+1/2} \psi_{-3/2}^* \psi_{-1/2}^* |0\rangle + \cdots,
\end{aligned}$$

where \cdots are irrelevant terms. Taking inner product with $|W\rangle$ one then gets:

$$\begin{aligned}
&\tau_W(\{y^{-1}\} - \{x^{-1}\}) \\
&= 1 + \sum_{j \geq 1} a_j \frac{(-x)^{-3j-1} - y^{-3j-1}}{(-x)^{-1} - y^{-1}} - \sum_{k \geq 1} b_k \frac{(-x)^{-3k} y^{-1} - (-x)^{-1} y^{-3k}}{(-x)^{-1} - y^{-1}} \\
&+ \sum_{j \geq k \geq 1} a_j b_k \cdot \frac{(-x)^{-3j-1} y^{-3k} - (-x)^{-3k} y^{-3j-1}}{(-x)^{-1} - y^{-1}} \\
&- \sum_{1 \leq j < k} a_j b_k \cdot \frac{(-x)^{3k} y^{-3j-1} - (-x)^{-3j-1} y^{-3k}}{(-x)^{-1} - y^{-1}} \\
&= \sum_{j, k \geq 0} a_j b_k \cdot \frac{(-x)^{-3j-1} y^{-3k} - (-x)^{-3k} y^{-3j-1}}{(-x)^{-1} - y^{-1}} \\
&= \frac{a(-x) \cdot b(y) - a(y) b(-x)}{x + y}.
\end{aligned}$$

Note we have obtained

$$(290) \quad \tau_W(\{x^{-1}\} + \{y^{-1}\}) = \sum_{j, k \geq 0} a_j b_k \cdot \frac{x^{-3j-1} y^{-3k} - x^{-3k} y^{-3j-1}}{x^{-1} - y^{-1}},$$

which is a formula that has been used in Okounkov [20] to prove the Witten Conjecture. The following are the first few terms:

$$\begin{aligned}
& \tau_W(\{x^{-1}\} + \{y^{-1}\}) = \frac{1}{x^{-1} - y^{-1}} \left((x^{-1} - y^{-1}) \right. \\
& + \left(\frac{5}{24}(x^{-4} - y^{-4}) - \frac{7}{24}(x^{-1}y^{-3} - x^{-3}y^{-1}) \right) \\
& + \left(\frac{385}{1152}(x^{-7} - y^{-7}) + \frac{5}{24} \cdot \frac{-7}{24}(x^{-4}y^{-3} - x^{-3}y^{-4}) \right. \\
& \quad \left. + \frac{-455}{1152}(x^{-1}y^{-6} - x^{-6}y^{-1}) \right) \\
& + \left(\frac{85085}{82944}(x^{-10} - y^{-10}) + \frac{385}{1152} \cdot \frac{-7}{24}(x^{-7}y^{-3} - x^{-3}y^{-7}) \right. \\
& \quad \left. + \frac{5}{24} \cdot \frac{-455}{1152} \cdot (x^{-4}y^{-6} - x^{-6}y^{-4}) + \frac{-95095}{82944}(x^{-1}y^{-9} - x^{-9}y^{-1}) \right) \\
& + \left(\frac{37182145}{7962624}(x^{-13} - y^{-13}) + \frac{85085}{82944} \cdot \frac{-7}{24}(x^{-10}y^{-3} - x^{-3}y^{-10}) \right. \\
& \quad + \frac{385}{1152} \cdot \frac{-455}{1152}(x^{-7}y^{-6} - x^{-6}y^{-7}) + \frac{5}{24} \cdot \frac{-95095}{82944}(x^{-4}y^{-9} - x^{-9}y^{-4}) \\
& \quad \left. + \frac{-40415375}{7962624}(x^{-1}y^{-12} - x^{-12}y^{-1}) \right) + \dots
\end{aligned}$$

Now we get:

$$\begin{aligned}
\sum_{m,n \geq 0} a_{m,n} x^{-m-1} y^{-n-1} &= -\frac{1}{x-y} + \frac{1}{x-y} \tau_W(\{y^{-1}\} - \{x^{-1}\}) \\
&= -\frac{1}{x-y} + \frac{a(-x) \cdot b(y) - a(y)b(-x)}{x^2 - y^2}.
\end{aligned}$$

This is exactly (282) if we interchange x with y and note

$$(291) \quad a_{m,n} = A_{n,m}.$$

The following are the first few terms of $A(x, y)$:

$$\begin{aligned}
A(x, y) &= \frac{5}{24xy^3} - \frac{7}{24x^2y^2} + \frac{5}{24x^3y} \\
&+ \frac{385}{1152xy^6} - \frac{455}{1152x^2y^5} + \frac{385}{1152x^3y^4} \\
&- \frac{385}{1152x^4y^3} + \frac{455}{1152x^5y^2} - \frac{385}{1152x^6y} \\
&+ \frac{85085}{82944xy^9} - \frac{95095}{82944x^2y^8} + \frac{85085}{82944x^3y^7} \\
&- \frac{43505}{41472x^4y^6} + \frac{45955}{41472x^5y^5} - \frac{43505}{41472x^6y^4} \\
&+ \frac{85085}{82944x^7y^3} - \frac{95095}{82944x^8y^2} + \frac{85085}{82944x^9y} + \dots
\end{aligned}$$

6.10. Bosonic N -point functions of topological 2D gravity. Let us conclude this paper by an application of the explicit expression of the Witten-Kontsevich tau-function in [29] and its geometric interpretation given by [4]. We combine these results with Theorem 5.3 to get:

Theorem 6.1. *For $F = \log Z_{WK}$,*

$$\begin{aligned}
(292) \quad & \sum_{j_1, \dots, j_n \geq 1} \frac{\partial^n F}{\partial T_{j_1} \cdots \partial T_{j_n}} \Big|_{\mathbf{T}=0} \xi_1^{-j_1-1} \cdots \xi_n^{-j_n-1} \\
&= (-1)^{n-1} \sum_{n\text{-cycles}} \prod_{i=1}^n \hat{A}(\xi_{\sigma(i)}, \xi_{\sigma(i+1)}),
\end{aligned}$$

where

$$(293) \quad \hat{A}(\xi_i, \xi_j) = \begin{cases} A(\xi_i, \xi_i), & \text{if } i = j, \\ \frac{1}{\xi_i - \xi_j} + A(\xi_i, \xi_j), & \text{if } i \neq j. \end{cases}$$

For a generalization of this result to Witten's r-spin curves, see [5].

Let us look at some special cases of (292). For $n = 1$, the prediction of (292) is

$$(294) \quad \sum_j \frac{\partial F}{\partial T_j} \Big|_{\mathbf{T}=0} \xi^{-j-1} = A(\xi, \xi).$$

By (282),

$$\begin{aligned}
A(\xi, \xi) &= \lim_{x \rightarrow \xi} \left(\frac{1}{x - \xi} + \frac{1}{x^2 - \xi^2} (a(x) \cdot b(-\xi) - a(-\xi) \cdot b(x)) \right) \\
&= \lim_{x \rightarrow \xi} \frac{1}{2x} (1 + a'(x) \cdot b(-\xi) - a(-\xi) \cdot b'(x)) \\
&= \frac{1}{2\xi} (1 + a'(\xi) \cdot b(-\xi) - a(-\xi) \cdot b'(\xi)).
\end{aligned}$$

By (58) one has

$$(295) \quad \left. \frac{\partial^n F}{\partial T_j} \right|_{\mathbf{T}=0} \xi^{-j-1} = \sum_{g \geq 1} \frac{(6g-3)!!}{24^g g! \xi^{6g+1}}.$$

So one gets an identity:

$$(296) \quad a'(\xi) \cdot b(-\xi) - a(-\xi) \cdot b'(\xi) = -1 + 2 \sum_{g \geq 1} \frac{(6g-3)!!}{24^g g! \xi^{6g}}.$$

It will be interesting to prove this directly by combinatorial method. For $n = 2$, the prediction of (292) is

$$(297) \quad \sum_{j,k} \left. \frac{\partial^2 F}{\partial T_j \partial T_k} \right|_{\mathbf{T}=0} \xi_1^{-j-1} \xi_2^{-k-1} = -\hat{A}(\xi_1, \xi_2) \hat{A}(\xi_2, \xi_1).$$

It is interesting to compare this with Dijkgraaf's formula for two-point function [16].

REFERENCES

- [1] M. Adler, P. van Moerbeke, *A matrix integral solution to two-dimensional W_p -gravity*. Comm. Math. Phys. 147 (1992), no. 1, 25-56.
- [2] M. Adler, T. Shiota, P. van Moerbeke, *Random matrices, Virasoro algebras, and noncommutative KP*. Duke Math. J. 94 (1998), no. 2, 379-431.
- [3] P.W.Andrson, *More is different*, Science 177 (172), No. 4047, 393-396.
- [4] F. Balogh, D. Yang, *Geometric interpretation of Zhou's explicit formula for the Witten-Kontsevich tau function*, arXiv:1412.4419.
- [5] F. Balogh, D. Yang, J. Zhou, *Explicit formula for Witten's r-spin partition function*, in preparation.
- [6] M. Bertola, B. Dubrovin, D. Yang, *Correlation functions of the KdV hierarchy and applications to intersection numbers over $\overline{\mathcal{M}}_{g,n}$* , arXiv:1504.06452.
- [7] B. Dubrovin, *Geometry of 2 D topological field theories*. Integrable systems and quantum groups (Montecatini Terme, 1993), 120-348, Lecture Notes in Math., 1620, Springer, Berlin, 1996.
- [8] B. Dubrovin, Y. Zhang, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov - Witten invariants*, arXiv:math/0108160.
- [9] I.M. Gelfand, L.A. Dikii, *Asymptotic behaviour of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-de Vries equations*, Russian Math. Surveys 30:5 (1975), 77-113.

- [10] A.B. Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*. Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary. Mosc. Math. J. 1 (2001), no. 4, 551-568, 645.
- [11] C. Itzykson, J.-B. Zuber, *Combinatorics of the modular group II: the Kontsevich integrals*, Int. J. Mod. Phys. A7 (1992) 5661-5705.
- [12] V. Kac, A. Schwarz, *Geometric interpretation of the partition function of 2D gravity*. Phys. Lett. B 257 (1991), no. 3-4, 329-334.
- [13] M. E. Kazarian and S. K. Lando, *An algebro-geometric proof of Wittens conjecture*, J. Amer. Math. Soc. 20 (2007), 1079-1089.
- [14] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*. Comm. Math. Phys. **147** (1992), no. 1, 1-23.
- [15] R. B. Laughlin, *A different universe. Reinventing physics from the bottom down*. Basic Books, 2005.
- [16] K. Liu, H. Xu, *The n -point functions for intersection numbers on moduli spaces of curves*, Adv. Theor. Math. Phys. 15 (2011), 1201-1236.
- [17] E. Looijenga, *Intersection theory on Deligne-Mumford compactifications (after Witten and Kontsevich)*. Sminaire Bourbaki, Vol. 1992/93. Astrisque No. 216 (1993), Exp. No. 768, 4, 187-212.
- [18] I.G. MacDonald, *Symmetric functions and Hall polynomials*, 2nd edition. Clarendon Press, 1995.
- [19] T. Miwa, M. Jimbo, E. Date, *Solitons. Differential equations, symmetries and infinite-dimensional algebras*. Translated from the 1993 Japanese original by Miles Reid. Cambridge Tracts in Mathematics, 135. Cambridge University Press, Cambridge, 2000.
- [20] A. Okounkov, *Generating functions for intersection numbers on moduli spaces of curves*, Int. Math. Res. Not. 2002, no. 18, 933-957..
- [21] R. Pandharipande, A. Pixton, D. Zvonkine, *Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures*. arXiv: 1303.1043, 2013.
- [22] M. Sato, *Soliton Equations as Dynamical Systems on an Infinite Dimensional Grassmann Manifold*, RIMS Kokyuroku 439 (1981), 30-46.
- [23] G. Segal, G. Wilson, *Loop groups and equations of KdV type*. Inst. Hautes Études Sci. Publ. Math. No. 61 (1985), 5-65.
- [24] T. Shiota, *Characterization of Jacobian varieties in terms of soliton equations*. Invent. Math. 83 (1986), no. 2, 333-382.
- [25] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in Differential Geometry, vol.1, (1991) 243-310.
- [26] E. Witten, *Algebraic geometry associated with matrix models of two-dimensional gravity* (pp. 235-269). Topological methods in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX.
- [27] J. Zhou, *Intersection numbers on Deligne-Mumford moduli spaces and quantum Airy curve*, arXiv:1206.5896.
- [28] J. Zhou, *Topological recursions of Eynard-Orantin type for intersection numbers on moduli spaces of curves*. Lett. Math. Phys. 103 (2013), no. 11, 1191-1206.
- [29] J. Zhou, *Explicit formula for Witten-Kontsevich tau-function* arXiv:1306.5429.
- [30] J. Zhou, *Quantum deformation theory of the Airy curve and mirror symmetry of a point*, arXiv:1405.5296.

- [31] J. Zhou, *On absolute N -point function associated with Gelfand-Dickey polynomials*, preprint, 2015.
- [32] J. Zhou, *Emergent geometry, Frobenius manifolds, and quantum deformation theory*, in preparation.

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